OUTER ACTIONS OF A DISCRETE AMENABLE GROUP ON APPROXIMATELY FINITE DIMENSIONAL FACTORS III, THE TYPE III_λ CASE, $0 < \lambda < 1$, EXAMPLES

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Dedicated to the Memory of Masahiro Nakamura

Abstract. In this last article of the series on outer actions of a countable dicrete amenable group on AFD factors, we analyze outer actions of a countable discrete free abelian group on an AFD factor of type III_λ, $0 < \lambda < 1$, and compute outer conjugacy invariants. As a byproduct, we discover the asymmetrization technique for coboundary condition on a $\mathbb{T}$-valued cocycle of a torsion free abelian group, which might have been known by the group cohomologists. As the asymmetrization technique gives us a very handy criteria for coboundaries, we present it here in detail in the second section.

Contents

§0. Introduction.

This article concludes the series of our joint work, [KtT1],[KtT2], and [KtT3], on the outer conjugacy classification of outer actions of a countable discrete amenable group on an approximately finite dimensional, (abbreviated to AFD), factor by examining outer actions of a countable discrete abelian group G on an AFD factor $\mathcal{R}_\lambda$ of type III_λ, $0 < \lambda < 1$. The cocycle conjugacy classification theory of actions of a countable discrete amenable group on an AFD factor had been completed through the work of many mathematicians over three decades, [Cnn3, Cnn4, Cnn6, Cnn7, Jn, JT, Ocn,

KtST1, KtST2, KwST, ST1, ST2, ST3], prior to the outer conjugacy classification theory.

Unlike the general classification program in operator algebras, the outer conjugacy classification of a countable discrete amenable group on $\mathcal{R}_\lambda$ is almost smooth as shown in the series of previous work, see [KtT3]. Only non-smooth part of the classification theory stems from the classification of subgroups N of G: for instance the classification of subgroups of a torsion free abelian group of higher rank is non-smooth. We refer the work of Sutherland concerning Borel parameterization of polish groups, [St2]. When the modular automorphism part $N = \dot{\alpha}^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ of the outer action $\dot{\alpha}$ of G on $\mathcal{R}_\lambda$, is fixed, the set of invariants becomes a compact abelian group. It is a rare case in the theory of operator algebras. So we are encouraged to make a concrete analysis of outer conjugacy class of a countable discrete amenable group. Of course, without having a concrete date on the group G involved, we cannot make a fine analysis. So we take a countable discrete free abelian group G and study its outer actions on $\mathcal{R}_\lambda$ and identify the invariants completely. The justification of this restriction rests on the fact that all outer actions of a countable discrete abelian group A can be viewed as outer actions of G by pulling back the outer action via the quotient map: $G \mapsto A$. Thanks to all hard analytic work on the cocycle conjugacy classification in the past, cited in the reference, our work is very algebraic and indeed done by cohomological computations.

We will begin first by relating the discrete core of $\mathcal{R}_\lambda$ and the core of an AFD factor $\mathcal{R}_1$ of type III_1. This analysis will enable us to have a simple model construction with given invariants, which is presented here in Section 1. Single automorphisms and a pair of commuting automorphisms of $\mathcal{R}_\lambda$ are studied first. Then we will work on the asymmetrization of a cocycle of a countable discrete abelian group which will provide a powerful tool for analysis of the third cohomology group $\mathrm{H}^3(G, \mathbb{T})$. The general theory of group cohomology is available to us today, for example see [Brw]. But we have to work with individual cocycles to analyze outer actions. So we have to have a tool to work with a cocycle directly beyond the computation of the cohomology group. For example, we have to identify which data of a given cocycle contributes to the modular automorphism part of the action in question. Thus we will work on the cohomology group based on a very primitive method of chasing cocycles, through which we discover the asymmetrization technique which provides us a quite handy criterion for the coboundary condition on a cocycle of a torsion free abelian group. In our previous work, [KtT1, KtT2, KtT3], the outer conjugacy classification of a countable discrete amenable group outer actions were studied by a resolution of the relevant third cocycle. In the abelian case, it is shown that there is a universal resolution group which takes care of all third cocycles at once which simplifies greatly the investigation of outer actions of a countable discrete abelian group. The reduced modified HJR-sequence will provide us a tool to chase the cocycles along with the asymmetrization technique. The first step

of studying outer actions of countable discrete abelian group G on a factor $\mathcal{M}$ of type III_λ, $0 < \lambda < 1$, is to find a countable discrete amenable group H and a surjective homomorphism $\pi_G\colon H \mapsto G$ so that the pull back $\pi_G^*(c)$ is a coboundary, the process called the resolution of a cocycle $c \in \mathrm{Z}^3(G, \mathbb{T})$. Then the outer action $\dot\alpha$ is identified with a lifting $\mathfrak{s}_H^*(\alpha)$ of an action α of H through a cross-section $\mathfrak{s}_H : G \mapsto H$ of the homomorphism π_G. Luckily, a countable discrete abelian group G admits a universal resolution $\{H, \pi_G\}$, a group H and a surjective homomorphism $\pi_G\colon H \mapsto G$ such that $\pi_G^*\big(\mathrm{H}^3(H, \mathbb{T})\big) = \{1\}$. The group H is constructed via relatively simple process from a countable discrete free abelian group G. This makes us possible to reduce the study of an outer action $\dot\alpha$ of G to that of an action α of H. Now, the action α of H does not lift to the discrete core $\widetilde{\mathcal{M}}_d$ if $\mathrm{mod}(\alpha) \neq 1$. So we construct a central extension H_m of H:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n \to z_0^n} H_\mathrm{m} \longrightarrow H \longrightarrow 1$$

and work with the characteristic cohomology group $\Lambda(H_\mathrm{m}, L, M, \mathbb{T})$ where the normal subgroup L stands for the inverse image $L = \pi_G^{-1}(N)$ with $N = \dot\alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$. Thus we are going to investigate the reduced modified HJR-sequence:

$$\begin{array}{ccccccc}
\mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{Res}} & \Lambda(H_\mathrm{m}, L, M, \mathbb{T}) & \xrightarrow{\delta} & \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) & \xrightarrow{\mathrm{Inf}} & \mathrm{H}^3(H,\mathbb{T}) \\
\Big\| & & \pi_\mathrm{m}^* \Big\downarrow & & \partial_{G_\mathrm{m}} \Big\downarrow & & \Big\| \\
\mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{res}} & \Lambda(H, M, \mathbb{T}) & \xrightarrow{\delta_{\mathrm{HJR}}} & \mathrm{H}^3(G,\mathbb{T}) & \xrightarrow{\pi_G^*} & \mathrm{H}^3(H,\mathbb{T})
\end{array}$$

Here $\mathfrak{s}$ is a fixed cross-section of the quotient map: $G \mapsto Q = G/N$. The groups appeared on the above exact sequences are all compact abelian groups and are indeed computable as shown in this paper.

We refer [Brw, EMc, McWh, Hb, Jn] for the general cohomology theory of abstract groups and [St1] for the cohomology theory related to von Neumann algebras. We refer [Tk1, Tk2, Tk3] for the general theory of von Neumann algebras. Concerning the discrete core of a factor of type III_λ, we refer [Cnn1, Cnn2, CT, FT1 and FT2].

This work was originated from the authors' visit to the Erwin Shrödinger Institute, Vienna, and the University of Rome, La Sapienza, in the spring of 2005 and further developed throughout the subsequent years. The second named author visited the Erwin Shrödinger Institute in the fall of 2008 again where the final touch on the joint work was given. The authors are greatly indebted to these institute, in particular to Professors Klaus Schmidt and Sergio Doplicher who made our collaboration possible and pleasant. We would like to record here our sincere appreciation to their support and hospitality.

§1 Simple Examples and Model Construction.

Factors of Type III_λ and Type III_1, and Their Cores: We begin by the following folklore theorem in the structure theory of factors of type III.

Theorem 1.1. *Let $\{\mathcal{M}_{0,1}, \tau, \theta\}$ be a factor of type II_∞ equipped with faithful semi finite normal trace τ and trace scaling automorphism θ by $\lambda, 0 < \lambda < 1$, i.e., $\tau\circ\theta = \lambda\tau$ and let $\mathcal{M} = \mathcal{M}_{0,1}^\theta$ be the fixed point subalgebra of $\mathcal{M}_{0,1}$ by θ. Then we have the following statements*:

i) *The von Neumann algebra $\mathcal{M}$ is a factor of type III_λ;*

ii) *The triplet $\{\mathcal{M}_{0,1}, \tau, \theta\}$ is conjugate to the discrete core of $\mathcal{M}$;*

iii) *For an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$, the following statements are equivalent*:

   a) $\alpha(\mathcal{M}) = \mathcal{M}$;

   b) $\alpha\circ\theta = \theta\circ\alpha$.

iv) *Let $\mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$ be the group of automorphisms of $\mathrm{Aut}(\mathcal{M}_{0,1})$ leaving $\mathcal{M}$ globally invariant. Then we have the following exact sequence*:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n\to\theta^n} \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M}) \xrightarrow{\alpha\to\alpha|_\mathcal{M}} \mathrm{Aut}(\mathcal{M}) \longrightarrow 1;$$

v) *The subgroup $\{\theta^n : n \in \mathbb{Z}\}$ is the Galois group of the pair $\{\mathcal{M}_{0,1}, \mathcal{M}\}$ in the sense that*

$$\{\theta^n : n \in \mathbb{Z}\} = \{a \in \mathrm{Aut}(\mathcal{M}_{0,1}) : \alpha(x) = x, x \in \mathcal{M}\}.$$

vi) *If $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$, then the modulus $\mathrm{mod}_{\mathcal{M}_{0,1}}(\alpha)$ as a member of $\mathrm{Aut}(\mathcal{M}_{0,1})$ gives the modulus $\mathrm{mod}_\mathcal{M}(\alpha)$ of the restriction $\alpha|_\mathcal{M} \in \mathrm{Aut}(\mathcal{M})$ in the following way*:

$$\mathrm{mod}_\mathcal{M}(\alpha) = \pi_{T'}\big(\mathrm{mod}_{\mathcal{M}_{0,1}}(\alpha)\big) \in \mathbb{R}/T'\mathbb{Z},$$

where

$$T' = -\log\lambda, \quad T = \frac{2\pi}{T'},$$
$$\pi_{T'} : s \in \mathbb{R} \mapsto \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}.$$

Proof. The statements (i) and (ii) are known in the general structure theory of a factor of type III, see [Tk2, Chapter XII, §2 and §6].

v) We prove the statement (v) first. Let ψ be a generalized trace of $\mathcal{M}$, i.e., a faithful semi-finite normal weight on $\mathcal{M}$ such that $\psi(1) = +\infty$ and $\sigma_T^\psi = \mathrm{id}$. Then the covariant system $\{\mathcal{M}_{0,1}, \theta\}$ is conjugate to the dual system $\left\{\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}/T\mathbb{Z}, \mathbb{Z}, \widehat{\sigma^\psi}\right\}$. So we identify them, so that $\mathcal{M}_{0,1}$ admits a periodic one parameter unitary group $\left\{u^\psi(s) : s \in \mathbb{R}\right\}$:

$$u^\psi(T) = 1, \quad \theta(u^\psi(s)) = \lambda^{is}u^\psi(s), \quad \text{and} \quad \sigma_s^\psi = \mathrm{Ad}(u^\psi(s))|_\mathcal{M}, \quad s \in \mathbb{R}.$$

Furthermore, the one parameter unitary group $\{u^\psi(s) : s \in \mathbb{R}\}$ together with $\mathcal{U}(\mathcal{M})$ generates the normalizer $\widetilde{\mathcal{U}}_0(\mathcal{M}) = \{v \in \mathcal{U}(\mathcal{M}_{0,1}) : v\mathcal{M}v^* = \mathcal{M}\}$, giving the semi-direct product decomposition $\widetilde{\mathcal{U}}_0 = \mathcal{U}(\mathcal{M}) \rtimes_{\sigma^\psi} \mathbb{R}/T\mathbb{Z}$. Suppose that $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$ leaves $\mathcal{M}$ pointwise fixed. We then show that $x \in \mathcal{M}$ and $u^\psi(s)^*\alpha\big(u^\psi(s)\big)$, $s \in \mathbb{R}$, commute:

$$\begin{aligned}
xu^\psi(s)^*\alpha\big(u^\psi(s)\big) &= u^\psi(s)^*u^\psi(s)xu^\psi(s)^*\alpha\big(u^\psi(s)\big)\\
&= u^\psi(s)^*\sigma_s^\psi(x)\alpha\big(u^\psi(s)\big) = u^\psi(s)^*\alpha\big(\sigma_s^\psi(x)u^\psi(s)\big)\\
&= u^\psi(s)^*\alpha\big(u^\psi(s)x\big) = u^\psi(s)^*\alpha\big(u^\psi(s)\big)x,
\end{aligned}$$

so that $u^\psi(s)^*\alpha\big(u^\psi(s)\big) = \mathcal{M}_{0,1} \cap \mathcal{M}' = \mathbb{C}$. Hence there exists a scalar $\mu(s) \in \mathbb{T}$ such that

$$\alpha\big(u^\psi(s)\big) = \mu(s)u^\psi(s), \quad s \in \mathbb{R}.$$

Since $u^\psi(T) = 1$, we have $\mu(T) = 1$. Since $\mu(s+t) = \mu(s)\mu(t)$, $s, t \in \mathbb{R}$, we have

$$\mu(s) = \lambda^{ins}, s \in \mathbb{R}, \quad \text{for some} \quad n \in \mathbb{Z}.$$

Since $\mathcal{M}$ together with $\{u^\psi(s) : s \in \mathbb{R}\}$ generate the whole algebra $\mathcal{M}_{0,1}$, we conclude that $\alpha = \theta^n$. This shows (v).

iii) Suppose that $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$ leave $\mathcal{M}$ globally invariant. Let $\beta = \alpha_\mathcal{M} = \alpha|_\mathcal{M}$ be the automorphism of $\mathcal{M}$ obtained as the restriction of α to $\mathcal{M}$. Then the uniqueness of a generalized trace on $\mathcal{M}$ gives a scalar $s \in \mathbb{R}$ and a unitary $v \in \mathcal{U}(\mathcal{M})$ such that

$$e^{-s}\psi = \psi{\circ}(\mathrm{Ad}(v){\circ}\beta).$$

This means that

$$\mathrm{mod}(\beta) = \mathrm{mod}(\mathrm{Ad}(v){\circ}\beta) = \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z},$$

and that σ^ψ and $\mathrm{Ad}(v){\circ}\beta$ commute. Hence it is possible to extend $\mathrm{Ad}(v){\circ}\beta$ to the automorphism $\gamma \in \mathrm{Aut}\{\mathcal{M}_{0,1}\}$ such that

$$\gamma\big(u^\psi(t)\big) = u^\psi(t), \quad t \in \mathbb{R}, \quad \gamma(x) = \mathrm{Ad}(v){\circ}\beta(x), \quad x \in \mathcal{M}.$$

Now we compare α and γ on $\mathcal{M}$:

$$\gamma(x) = \mathrm{Ad}(v){\circ}\beta(x) = \mathrm{Ad}(v){\circ}\alpha(x), \quad x \in \mathcal{M}.$$

From (v) it follows that α is of the form:

$$\alpha = \theta^n{\circ}\mathrm{Ad}(v^*){\circ}\gamma.$$

for some $n \in \mathbb{Z}$. Since θ commutes with both γ and $\mathrm{Ad}(v)$, α and θ commute. Hence the implication $(a) \Rightarrow (b)$ follows. The reversed implication: $(b) \Rightarrow (a)$ is trivial. So the proof of (iii) is complete.

iv) This follows from (iii) and (v).

Let $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \tau, \theta\right\}$ be the non-commutative flow of weights on $\mathcal{M}$ so that the covariant system $\{\mathcal{M}_{0,1}, \mathbb{Z}, \theta\}$ is identified with $\{\mathcal{M} \vee \{\psi\}, \theta_{T'}\}$

vi) Fix a member $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$ and let $\mathrm{m}(\alpha) = \mathrm{mod}(\alpha) \in \mathbb{R}$ so that

$$\tau{\circ}\alpha = e^{-\mathrm{m}(\alpha)}\tau.$$

Consider the crossed product

$$\widetilde{\mathcal{M}} = \mathcal{M}_{0,1} \rtimes_\theta \mathbb{Z} \cong \mathcal{M}\overline{\otimes}\mathcal{L}\left(\ell^2(\mathbb{Z})\right)$$

and the generalize trace $\varphi = \tau{\circ}\mathcal{E}$ on $\widetilde{\mathcal{M}}$:

$$\varphi(x) = \tau{\circ}\mathcal{E}(x) = \tau\left(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\mathrm{d}s\right), \quad x \in \widetilde{\mathcal{M}}_+.$$

With $U \in \mathcal{U}(\widetilde{\mathcal{M}})$ the unitary corresponding to the crossed product $\mathcal{M}_{0,1} \rtimes_\theta \mathbb{Z}$, we extend α to $\widetilde{\alpha} \in \mathrm{Aut}\left(\widetilde{\mathcal{M}}\right)$ by:

$$\widetilde{\alpha}(x) = \alpha(x), \quad x \in \mathcal{M}_{0,1}, \quad \widetilde{\alpha}(U) = U.$$

Then we have for each $x \in \widetilde{\mathcal{M}}_+$

$$\begin{aligned}
\varphi{\circ}\widetilde{\alpha}(x) &= \tau\left(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(\widetilde{\alpha}(x))\mathrm{d}s\right) = \tau\left(\alpha\left(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\mathrm{d}s\right)\right) \\
&= e^{-\mathrm{m}(\alpha)}\tau\left(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\mathrm{d}s\right) \\
&= e^{-\mathrm{m}(\alpha)}\varphi(x).
\end{aligned}$$

Hence we get

$$\mathrm{mod}(\widetilde{\alpha}) = [\mathrm{m}(\alpha)]_{T'} = \mathrm{m}(\alpha) + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}. \tag{1.1}$$

Since the covariant systems $\{\mathcal{M}, \alpha\}$ and $\left\{\widetilde{\mathcal{M}}, \widetilde{\alpha}\right\}$ are cocycle conjugate, we have $\mathrm{mod}(\widetilde{\alpha}) = \mathrm{mod}(\alpha)$. This completes the proof. ♡

Now, we denote by $\mathcal{R}_0$ an approximately finite dimensional factor of type II_1 throughout the paper.

A factor $\mathcal{M}_1$ of type III_1 generates one parameter family $\{\mathcal{M}_\lambda : 0 < \lambda \leq 1\}$ of factors of type III_λ, who share the same discrete core $\mathcal{M}_{0,1}$. So let $\mathcal{M}_1$ be a factor of type III_1, and $\{\mathcal{M}_{0,1}, \theta_s, s \in \mathbb{R}\}$ be the non-commutative flow of weights on $\mathcal{M}_1$, i.e., $\mathcal{M}_{0,1}$ is a factor of type II_∞ equipped with a trace scaling one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ and a faithful semi-finite normal trace τ such that

$$\mathcal{M}_1 = \mathcal{M}_{0,1}^\theta, \quad \tau{\circ}\theta_s = e^{-s}\tau, \ s \in \mathbb{R}.$$

The following is a folklore theorem in the structure theory of type III.

Theorem 1.2. *In the above context, fixing $T' > 0$, set*

$$\lambda = e^{-T'}, \quad T = \frac{2\pi}{T'},$$

and let $\mathcal{M}_\lambda$ be the fixed point subalgebra $\mathcal{M}_{0,1}^{\theta_{T'}}$ of $\mathcal{M}_{0,1}$ under the automorphism $\theta_{T'}$. Then the following statements hold:

i) *The subalgebra $\mathcal{M}_\lambda \subset \mathcal{M}_{0,1}$ is a factor of type III_λ, whose discrete core is conjugate to the pair $\{\mathcal{M}_{0,1}, \theta_{T'}\}$.*

ii) *The triplet $\{\mathcal{M}_{0,1}, \mathcal{M}_\lambda, \theta_{T'}\}$ is a Galois triplet in the sense*:

$$\mathrm{Gal}(\mathcal{M}_{0,1}/\mathcal{M}_\lambda) = \{\theta_{T'}^n : n \in \mathbb{Z}\},$$

where

$$\mathrm{Gal}(\mathcal{M}/\mathcal{N}) = \{\alpha \in \mathrm{Aut}(\mathcal{M}) : \alpha|_{\mathcal{N}} = \mathrm{id}\}$$

for any pair $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras. Furthermore, we have the following exact sequence:

$$1 \longrightarrow \{\theta_{T'}^n : n \in \mathbb{Z}\} \longrightarrow \mathrm{Aut}(\mathcal{M}_\lambda)_{\mathrm{m}} \longrightarrow \mathrm{Aut}(\mathcal{M}_\lambda) \longrightarrow 1$$

and

$$\begin{aligned} \mathrm{Aut}(\mathcal{M}_\lambda)_{\mathrm{m}} &= \{\widetilde{\alpha} \in \mathrm{Aut}(\mathcal{M}_{0,1}) : \widetilde{\alpha}(\mathcal{M}_\lambda) = \mathcal{M}_\lambda\} \\ &= \{\widetilde{\alpha} \in \mathrm{Aut}(\mathcal{M}_{0,1}) : \widetilde{\alpha}\circ\theta_{T'} = \theta_{T'}\circ\widetilde{\alpha}\}. \end{aligned}$$

iii) *Another pair $\{\mathcal{M}_\lambda, \mathcal{M}_1\}$ forms a Galois pair*:

$$\mathrm{Gal}(\mathcal{M}_\lambda/\mathcal{M}_1) = \big\{\theta_{\dot{s}_{T'}} : \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}, s \in \mathbb{R}\big\},$$

i.e., an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ is of the form $\alpha = \theta_{\dot{s}_{T'}}$ for some $\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}$ if and only if $\alpha(x) = x, x \in \mathcal{M}_1$.

iv) *The modulus of $\theta_{\dot{s}_{T'}} \in \mathrm{Aut}(\mathcal{M}_\lambda)$ is precisely $-\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}$ itself, i.e.,*

$$\mathrm{mod}\big(\theta_{\dot{s}_{T'}}\big) = -\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}.$$

If any one of $\mathcal{M}_\lambda, \mathcal{M}_1$ and $\mathcal{M}_{0,1}$ is approximately finite dimensional, then all others are approximately finite dimensional and in addition the following statements hold:

v) *If $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ has aperiodic modulus $\mathrm{m} = \mathrm{mod}(\alpha)$, i.e., if $k\mathrm{m} \neq 0$ for every non-zero integer $k \in \mathbb{Z}$, or equivalently if*

$$\frac{\{\mathrm{mod}(\alpha)\}_{T'}}{T'} \notin \mathbb{Q},$$

then α is cocycle conjugate to $\theta_{-\mathrm{m}}$.

vi) *If an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ has trivial asymptotic outer period, i.e., $p_a(\alpha) = 0$, then its cocycle conjugacy class is determined by its modulus $\mathrm{m} = \mathrm{mod}(\alpha) \in \mathbb{R}/T'\mathbb{Z}$. In fact, the automorphism α is cocycle conjugate to the automorphism $\theta_{-\mathrm{m}}\otimes\sigma_0$ on $\mathcal{M}_\lambda \cong \mathcal{M}_\lambda\overline{\otimes}\mathcal{R}_0$, where $\sigma_0 \in \mathrm{Aut}(\mathcal{R}_0)$ is any aperiodic automorphism of the approximately finite dimensional factor $\mathcal{R}_0$. If $\mathrm{m} \neq 0$, then we have $\theta_{\mathrm{m}} \sim \theta_{\mathrm{m}}\otimes\sigma_0$.*

Proof. We present a proof the statements (v) and (vi). Choose an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ such that $\mathrm{m} = \mathrm{mod}(\alpha)$ is aperiodic. Let $\mathcal{R}_0$ be an approximately finite dimensional (to be abbreviated to AFD afterward) factor of type II_1 realized as the infinite tensor product of two by two matrix algebras

$$\mathcal{R}_0 = \prod_{n\in\mathbb{Z}}{}^{\otimes}\{M_n, \tau_n\}$$

relative to the normalized traces $\tau_n = \mathrm{Tr}/2$ on $M_n = \mathrm{M}(2, \mathbb{C})$. Let σ_0 be the Bernouille shift automorphism of $\mathcal{R}_0$, i.e., the automorphism determined by the following:

$$\sigma_0\left(\prod_{n\in\mathbb{Z}}{}^{\otimes} x_n\right) = \prod_{n\in\mathbb{Z}}{}^{\otimes} x_{n+1}.$$

Then thanks to the grand theorem of Connes, [Cnn6, Tk3, page 267], α and $\alpha \otimes \sigma_0$ are cocycle conjugate under the identification of $\mathcal{M}_\lambda$ and $\mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0$ because the asymptotic outer period $p_a(\alpha)$ of α is zero, $p_a(\alpha) = 0$. The same is true for θ_{m}, i.e., $\theta_{\mathrm{m}} \sim_c \theta_{\mathrm{m}} \otimes \sigma_0$, where "$\sim_c$" means the outer conjugacy. Since $\mathrm{mod}(\alpha_1 \otimes \alpha_2) = \mathrm{mod}(\alpha_1) + \mathrm{mod}(\alpha_2)$ on $\mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda$, we have

$$\alpha \sim_c \alpha \otimes \sigma_0 \sim_c \alpha \otimes \theta_{\mathrm{m}} \otimes \theta_{-\mathrm{m}} \sim_c \sigma_0 \otimes \theta_{-\mathrm{m}} \sim_c \theta_{-\mathrm{m}}.$$

This proves the statement (v).

vi) Suppose that $p \in \mathbb{N}$ is the period of $\mathrm{m} \in \mathbb{R}/T'\mathbb{Z}$, i.e., the smallest non-negative integer with $p\mathrm{m} = 0$. We assume that $p \neq 0$. Let

$$\{e_{j,k} : 1 \le j, k \le p\}$$

be the standard matrix units of the $p \times p$-matrix algebra $\mathrm{M}(p; \mathbb{C})$, and for each $n \in \mathbb{N}$ set $M_n = \mathrm{M}(p; \mathbb{C}), n \in \mathbb{N}$, and also consider the diagonal unitary

$$u_n = \sum_{i=1}^{p} \exp\left(2\pi\mathrm{i}\left(\frac{i-1}{p}\right)\right) e_{i,i} \in \mathrm{U}(p; \mathbb{C}) \subset M_n$$

of order p, i.e., $u_n^p = 1$. Now we identify the AFD factor $\mathcal{R}_0$ with the infinite tensor product:

$$\mathcal{R}_0 = \prod_{n\in\mathbb{N}}{}^{\otimes}\{M_n, \tau_n\}, \quad \tau_n = \frac{1}{n}\mathrm{Tr}$$

and let

$$\sigma_p = \prod_{n\in\mathbb{N}}{}^{\otimes}\mathrm{Ad}(u_n) \in \mathrm{Aut}(\mathcal{R}_0) \in \mathrm{Aut}(\mathcal{R}_0).$$

Then the automorphism σ_p has the properties:

$$\begin{gathered}
\sigma_p^k \notin \mathrm{Int}(\mathcal{R}_0),\ \text{for } k = 1, \cdots, p-1, \text{ and } \sigma_p^p = \mathrm{id} \\
\theta_{\mathrm{m}} \sim_c \theta_{\mathrm{m}} \otimes \sigma_p \quad \text{on} \quad \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0, \\
\theta_{\mathrm{m}} \otimes \theta_{-\mathrm{m}} \sim_c \mathrm{id} \otimes \mathrm{id} \otimes \sigma_p \quad \text{on} \quad \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0, \\
\sigma_0 \otimes \sigma_p \sim_c \sigma_0 \quad \text{on} \quad \mathcal{R}_0 \mathbin{\overline{\otimes}} \mathcal{R}_0 \cong \mathcal{R}_0.
\end{gathered}$$

If $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ has the trivial asymptotic outer period $p_a(\alpha) = 0$, then the automorphism α has the properties:

$$\begin{gathered}
\alpha \sim_c \sigma_p \otimes \alpha \quad \text{on} \quad \mathcal{M}_\lambda \cong \mathcal{R}_0 \overline{\otimes} \mathcal{M}_\lambda, \\
\theta_{\mathrm{m}} \otimes \alpha \sim_c \mathrm{id} \otimes \sigma_0 \quad \text{on} \quad \mathcal{M}_\lambda \overline{\otimes} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \overline{\otimes} \mathcal{R}_0, \\
\theta_{-\mathrm{m}} \otimes \sigma_0 \sim_c \theta_{-\mathrm{m}} \otimes \theta_{\mathrm{m}} \otimes \alpha \sim_c \sigma_p \otimes \alpha \sim_c \alpha
\end{gathered}$$

under the isomorphisms:

$$\mathcal{M}_\lambda \overline{\otimes} \mathcal{R}_0 \cong \mathcal{M}_\lambda \overline{\otimes} \mathcal{M}_\lambda \overline{\otimes} \mathcal{M}_\lambda \cong \mathcal{R}_0 \overline{\otimes} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda.$$

This completes the proof. ♡

Thus if $\mathrm{mod}(\alpha)$ is aperiodic, or $p_a(\alpha) = 0$, then the grand theorem of Connes [Cnn6, Tk3, page 270], identifies the cocycle conjugacy class of $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$. But if $\mathrm{mod}(\alpha)$ has non trivial period, and $p_1 = p_a(\alpha) \neq 0$, then the cocycle conjugacy class of α involves algebraic invariants. For example, one has to consider the extension of α to the discrete core $\widetilde{\mathcal{M}}_{\lambda,d}$ on which α alone cannot act. In fact, one has to consider a larger group $\mathbb{Z}^2$ than the integer group $\mathbb{Z}$. So we continue to the next paragraph.

Invariants for Single Automorphisms: We consider a single automorphism of a factor $\mathcal{M}$ of type III_λ, which can be viewed as an action of the integer additive group $\mathbb{Z}$. As the integer group $\mathbb{Z}$ appears in many different roles, we denote it by $G = \mathbb{Z}$. Let a_1 be the generator of the group G so that $G = \mathbb{Z}a_1$. Sometime, we view G as a multiplicative group in which case G becomes $G = \left\{a_1^k : k \in \mathbb{Z}\right\}$. Since the integer group is cohomologically trivial, i.e., $\mathrm{H}^2(G, \mathbb{T}) = \mathrm{H}^3(G, \mathbb{T}) = \{1\}$, there is no distinction between the cocycle conjugacy problem and the outer conjugacy problem of actions of G. Namely, an outer action $\dot{\alpha}$ of G comes always from an action α of G and outer conjugacy of the outer action $\dot{\alpha}$ of G is the same as the cocycle conjugacy of the action α of G. Hence the obstruction $\mathrm{Ob}(\dot{\alpha})$ of $\dot{\alpha}$ and the characteristic invariant $\chi(\alpha)$ of α is handily identified. The same is true for the modular obstruction $\mathrm{Ob}_{\mathrm{m}}(\dot{\alpha})$ and the modular characteristic invariant $\chi_{\mathrm{m}}(\alpha)$.

As the single automorphism cocycle conjugacy classification wasn't handled properly in our previous work, [KtST1, KtST2], and more importantly the presentation of a single automorphism on a factor of type III_λ in the book of the second named author [Tk3] contains a minor mistake, we present it here in some detail.

Since the case that the modulus $\mathrm{m} = \mathrm{mod}(\alpha)$ is aperiodic, then the last theorem takes care of the cocycle conjugacy of α, i.e., it must be cocycle conjugate to $\theta_{-\mathrm{m}}$. So we handle only the case that $\{\mathrm{mod}(\alpha)\}_{T'}$ is rational multiple of T'.

Suppose $\alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})) = \mathbb{Z}b_1$ and $b_1 = p_1 a_1, p_1 \in \mathbb{N}$.

Choose a pair $p_1, q_1 \in \mathbb{N}$ of positive integers $q_1 < p_1$ such that

$$\mathrm{m} = \frac{q_1}{p_1} T' + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}, \quad 0 \leq q_1 < p_1.$$

Then we form a group extension:

$$\begin{gathered} G_{\mathrm{m}} = \{(g, s) \in G \times \mathbb{R} : g\mathrm{m} = \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}\}, \\ 0 \longrightarrow \mathbb{Z} \xrightarrow{k \to (0, kT')} G_{\mathrm{m}} \xrightarrow{\mathrm{pr}_1} G \longrightarrow 0. \end{gathered} \tag{1.2}$$

Set

$$\begin{gathered} z_0 = (0, T'), \quad z_1 = (a_1, \{\mathrm{m}\}_{T'}), \\ b_1 = p_1 z_1 - q_1 z_0, \quad N = \mathbb{Z} b_1, \quad Q_{\mathrm{m}} = G_{\mathrm{m}}/N. \end{gathered} \tag{1.3}$$

The group G_{m} is equipped with a distinguished homomorphism $\mathrm{k}_{\mathrm{m}} = \mathrm{pr}_2$ to $\mathbb{R}$:

$$\mathrm{k}_{\mathrm{m}}(g, s) = s \in \mathbb{R}, \quad (g, s) \in G_{\mathrm{m}}. \tag{1.4}$$

Let $\pi_Q : g \in G_{\mathrm{m}} \mapsto \dot{g} \in Q_{\mathrm{m}}$ be the quotient map and further set

$$D_1 = \gcd(p_1, q_1), \quad \text{and} \quad r_1 = \frac{p_1}{D_1}, \quad s_1 = \frac{q_1}{D_1}, \tag{1.5}$$

and find a pair $u_1, v_1 \in \mathbb{Z}$ of integers such that

$$1 = r_1 u_1 - s_1 v_1, \quad \text{equivalently} \quad D_1 = p_1 u_1 - q_1 v_1,$$

which can be done through the Euclid algorithm. In the event that $q_1 = 0$, the modulus m is trivial, i.e., $\mathrm{m} = 0$ and $G_{\mathrm{m}} = G \oplus \mathbb{Z}$.

Theorem 1.3 (Invariants for a Single Automorphism with Periodic Modulus). *In the case that p_1 and q_1 are both non-zero, we have the following statements with $D_1 = \gcd(p_1, q_1)$:*

i) *The pair $\{z_0, z_1\}$ is a free basis of G_{m} so that every element $g \in G_{\mathrm{m}}$ is written uniquely in the form:*

$$g = e_0(g) z_0 + e_1(g) z_1.$$

ii) *The group G_{m} admits another free basis $\{w_0, w_1\}$ such that*

$$b_1 = D_1 w_1,$$

and therefore

$$\begin{gathered} N = D_1 \mathbb{Z} w_1, \quad Q_{\mathrm{m}} = \mathbb{Z}\dot{w}_0 \oplus \mathbb{Z}\dot{w}_1, \\ D_1 \dot{w}_1 = 0 \ \ \text{in} \ \ Q_{\mathrm{m}} \cong \mathbb{Z} \oplus \mathbb{Z}_{D_1}, \end{gathered}$$

where the dotted notations indicate their images in the quotient group Q_{m}.

iii) *The character group* $\widehat{Q_{\mathrm{m}}}$ *of* Q_{m} *and the characteristic cohomology group* $\Lambda(G_{\mathrm{m}}, N, \mathbb{T})$ *is identified under the correspondence*:

$$\lambda_\chi(nb_1; g) = \chi(\pi_Q(g))^n, \quad g \in G_{\mathrm{m}},\ \chi \in \widehat{Q_{\mathrm{m}}}. \tag{1.6}$$

iv) *The character group* $\widehat{Q_{\mathrm{m}}}$ *is given by the exact sequence*:

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{R} \oplus \left(\frac{1}{D_1}\mathbb{Z}\right) \xrightarrow{\exp(2\pi\mathrm{i}\ \cdot)} \mathbb{T} \oplus \mathbb{Z}_{D_1} = \widehat{Q_{\mathrm{m}}} \longrightarrow 0,$$

which describes the characteristic cohomology group $\Lambda(G_{\mathrm{m}}, N, \mathbb{T})$:

$$\Lambda(G_{\mathrm{m}}, N, \mathbb{T}) \cong \mathbb{T} \oplus \mathbb{Z}_{D_1}. \tag{1.7}$$

If $\chi(z_0)$ *is a root of unity, then the outer period* $p_{\mathrm{o}}(\alpha)$ *of* α *is given as the product* $p_1 s_{\mathrm{o}}$ *with* $s_{\mathrm{o}} \in \mathbb{Z}_+$ *the smallest non-negative integer* $s \in \mathbb{Z}_+$ *such that* $1 = \chi(z_0)^s$. *If* $\chi(z_0)$ *is not a root of unity, then the corresponding automorphism* α *is aperiodic, i.e.,* $p_{\mathrm{o}}(\alpha) = 0$.

Proof. i) Since $\mathrm{pr}_1(z_1) = a_1$ and G is a free abelian group, the exact sequence (1.2) splits along with the cross-section: $m \in G \mapsto mz_1 \in G_{\mathrm{m}}$.

ii) We set

$$w_0 = u_1 z_0 - v_1 z_1, \quad w_1 = -s_1 z_0 + r_1 z_1.$$

Since

$$z_0 = r_1 w_0 + v_1 w_1, \quad z_1 = s_1 w_0 + u_1 w_1,$$

the pair $\{w_0, w_1\}$ is a free basis of G_{m} such that

$$G_{\mathrm{m}} = \mathbb{Z}w_0 + \mathbb{Z}w_1, \quad b_1 = D_1 w_1, \quad N = D_1\mathbb{Z}w_1,$$
$$Q_{\mathrm{m}} = G_{\mathrm{m}}/N = \mathbb{Z}\dot{w}_0 \oplus \mathbb{Z}\dot{w}_1,$$

as we wanted.

iii) Since $\mathrm{H}^2(N, \mathbb{T}) = \{1\}$, the second cocycle part of a characteristic cocycle in $\mathrm{Z}(G_{\mathrm{m}}, N, \mathbb{T})$ is taken to be trivial, so that the λ-part vanishes on N and therefore it is a character of G_{m} which vanishes on N and factors through the quotient map $\pi_Q : G_{\mathrm{m}} \mapsto Q_{\mathrm{m}}$. Thus it is of the form:

$$\lambda(b_1; g) = \chi(\pi_Q(g)), \quad g \in G_{\mathrm{m}},\ \chi \in \widehat{Q_{\mathrm{m}}}.$$

iv) It follows from (ii) that the character group $\widehat{Q_{\mathrm{m}}}$ is parameterized by $\mathbb{R} \oplus \left(\frac{1}{D_1}\mathbb{Z}\right)$:

$$\chi_{x,y}(g) = \exp(2\pi\mathrm{i}(xf_0(g) + yf_1(g))), \quad g = f_0(g)w_0 + f_1(g)w_1 \in G_{\mathrm{m}},$$

with $(x, y) \in \mathbb{R} \oplus \left(\frac{1}{D_1}\mathbb{Z}\right)$. This gives the exact sequence:

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{R} \oplus \left(\frac{1}{D_1}\mathbb{Z}\right) \xrightarrow{(x,y)\to\chi_{x,y}} \widehat{Q_{\mathrm{m}}} = \mathbb{T} \oplus \mathbb{Z}_{D_1} \longrightarrow 0$$

The assertion (iv) follows. This completes the proof. ♡

Model Construction: Let G be a fixed countable discrete amenable group and $\{H, \pi_G\}$ be a universal resolution group of the third cocycles of G, i.e, $\pi_G : H \mapsto G$ is a surjective homomorphisms such that

$$\pi_G^*\big(\mathrm{Z}^3(G, \mathbb{T})\big) \subset \mathrm{B}^3(H, \mathbb{T}).$$

We require H to be a countable discrete amenable group. Let $M = \mathrm{Ker}(\pi_G)$. Fix a normal subgroup N of G and set $L = \pi_G^{-1}(N)$. With a fixed invariant homomorphism $\mathrm{m} \in \mathrm{Hom}_G(N, \mathbb{R}/T'\mathbb{Z})$ such that $\mathrm{Ker}(\mathrm{m}) \supset N$, we use the notation m for $\mathrm{m}\circ\pi_G$ for short and form a group extension H_m:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_\mathrm{m} \xrightarrow{\pi_\mathrm{m}} H \longrightarrow 1,$$

where

$$\begin{gathered} H_\mathrm{m} = \{(g, s) \in H \times \mathbb{R} : \mathrm{m}(g) = \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}\}, \\ \pi_\mathrm{m}(g, s) = g \in H, \quad \mathrm{k}(g, s) = s \in \mathbb{R}, \quad (g, s) \in H_\mathrm{m}. \end{gathered}$$

Then we get the following reduced modified HJR-sequence:

$$\cdots \longrightarrow \mathrm{H}^2(H, \mathbb{T}) \xrightarrow{\mathrm{Res}} \Lambda(H_\mathrm{m}, L, M, \mathbb{T}) \xrightarrow{\delta} \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) \longrightarrow 1$$

Thus every modular obstruction cocycle $(c, \nu) \in \mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ is of the form:

$$(c, \nu) \equiv \delta(\lambda, \mu) \mod \mathrm{B}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}).$$

Consequently the construction of an outer action $\dot\alpha$ of G on an AFD factor $\mathcal{M}_\lambda$ of type III_λ with $\mathrm{Ob}_\mathrm{m}(\dot\alpha) = ([c], \nu) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ is reduced to the construction of an action $\alpha^{\lambda,\mu}$ of H_m such that

$$\begin{gathered} \big(\alpha^{\lambda,\mu}\big)^{-1}(\mathrm{Int}(\mathcal{M}_\lambda)) \supset M, \quad \big(\alpha^{\lambda,\mu}\big)^{-1}(\mathrm{Cnt}(\mathcal{M}_\lambda)) = L, \\ \chi\big(\alpha^{\lambda,\mu}\big) = [\lambda, \mu] \in \Lambda(H_\mathrm{m}, L, M, \mathbb{T}), \\ \mathrm{mod}\big(\alpha^{\lambda,\mu}_g\big) = \mathrm{m}(\pi_G(g)), \quad g \in H_\mathrm{m}. \end{gathered}$$

So fix a set of invariants $(\lambda, \mu) \in \mathrm{Z}(H_\mathrm{m}, L, M, \mathbb{T})$ and $\mathrm{m} \in \mathrm{Hom}_G(G, \mathbb{R}/T'\mathbb{Z})$ such that $\mathrm{Ker}(\mathrm{m}) \supset N$. We are going to construct the model action $\alpha^{\lambda,\mu}$ of H_m as follows:

Step I: Let X be a countable but infinite set on which H_m acts freely from the left. In the case that H_m is an infinite group, then we take X to be H_m itself and let H_m act on it by the multiplication from the left. So the infinite set X is only needed when H_m is a finite group in which case X can be taken to be the product set $X = H_\mathrm{m} \times \mathbb{N}$ and H_m act on the first component by the left multiplication . Let $\{M_x, x \in H_\mathrm{m}\}$ be the set of 2 by 2 matrix algebras $\mathrm{M}(2, \mathbb{C})$ indexed by elements $x \in X$.

Step II: Form the infinite tensor product

$$\mathcal{R}_0 = \prod_{x \in X}^{\otimes} \{M_x, \tau_x\}$$

relative to the normalized trace

$$\tau_x \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2}.$$

Then let σ^0 be the Bernouille action of H_{m} on $\mathcal{R}_0$ which is determined by:

$$\sigma_g^0 \left(\prod_{x \in X}^{\otimes} a_x \right) = \prod_{x \in X}^{\otimes} a_{gx}.$$

Step III: Form the twisted partial crossed product of $\mathcal{R}_0$ by N relative to the second cocycle $\mu \in \mathrm{Z}^2(N, \mathbb{T})$ and the action σ^0:

$$\mathcal{M}_0 = \mathcal{R}_0 \rtimes_{\sigma^0, \mu} N.$$

Let $\{U(m) : m \in N\}$ be the projective unitary representation of N to $\mathcal{M}_0$ corresponding to the twisted crossed product so that

$$\begin{aligned} U(g)U(h) &= \mu(g; h)U(gh), \quad g, h \in N; \\ U(g)aU(g)^* &= \sigma_g^0(a), \quad a \in \mathcal{R}_0, g \in N. \end{aligned}$$

Let $\sigma^{\lambda,\mu}$ be the action of H_{m} on $\mathcal{M}_0$ determined by:

$$\begin{aligned} \sigma_g^{\lambda,\mu}(U(m)) &= \lambda(gmg^{-1}; g)U(gmg^{-1}), \quad m \in N, g \in H_{\mathrm{m}}; \\ \sigma_g^{\lambda,\mu}(a) &= \sigma_g^0(a), \quad a \in \mathcal{R}_0, g \in H_{\mathrm{m}}. \end{aligned}$$

Step IV: Let $\mathcal{M}_{0,1}$ be the AFD factor of type II_∞ equipped with trace scaling one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ and set

$$\mathcal{R}_{0,1} = \mathcal{M}_{0,1} \overline{\otimes} \mathcal{M}_0.$$

We then set the action $\widetilde{\alpha}^{\lambda,\mu}$ to be the following:

$$\tilde{\alpha}_g^{\lambda,\mu} = \theta_{\mathrm{m}(g)} \otimes \sigma_g^{\lambda,\mu} \text{ on } \mathcal{R}_{0,1}, \quad g \in H_{\mathrm{m}}.$$

Set

$$\mathcal{R} = (\mathcal{R}_{0,1})^{\widetilde{\alpha}_{z_0}}.$$

Since the automorphism $\widetilde{\alpha}_{z_0} = \theta_{T'} \otimes \sigma_{z_0}^{\lambda,\mu}$ scales the trace τ by $\lambda = e^{-T'}$, the von Neumann algebra $\mathcal{R}$ is an AFD factor of type III_λ. Finally we define the action $\alpha^{\lambda,\mu}$ by the following:

$$\alpha_g^{\lambda,\mu} = \widetilde{\alpha}_g^{\lambda,\mu}\Big|_{\mathcal{R}}, \quad g \in H,$$

which makes sense because $\widetilde{\alpha}_{z_0}$ acts trivially on $\mathcal{R}$.

Theorem 1.4 (Model Action). i) *The action $\alpha = \alpha^{\lambda,\mu}$ constructed above has the invariants*:

$$N = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}_\lambda)), \quad \mathrm{mod}\,(\alpha_g) = \mathrm{m}(g), \quad g \in H,$$
$$\chi(\alpha) = [\lambda, \mu] \in \Lambda(H_\mathrm{m}, L, M, \mathbb{T}),$$
$$\nu_\alpha(g) = \left[\frac{T\mathrm{Log}(\lambda(g; z_0))}{2\pi}\right]_T \in \mathbb{R}/T\mathbb{Z}, \quad g \in N.$$

ii) *Let $\mathfrak{s}_H : G \mapsto H$ be a cross-section of the homomorphism $\pi_G : H \mapsto G$. Then the outer action $\alpha^{\lambda,\mu}_{\mathfrak{s}_H}$ of G has the associated modular obstruction $\delta([\lambda, \mu]) = [c^{\lambda,\mu}, \nu^\lambda] \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$.*

The construction of (i) *and* (ii) *exhausts all outer actions of G on the approximately finite dimensional factor $\mathcal{R}$ of type III_λ up to outer conjugacy.*

Proof. i) Let $\widetilde{\alpha}$ denote the action $\widetilde{\alpha}^{\lambda,\mu}$ of H_m on $\mathcal{R}_{0,1}$. Since $\mathcal{R}$ is the fixed point subalgebra of $\mathcal{R}_{0,1}$ under the automorphism $\widetilde{\alpha}_{z_0}$, the restriction $\alpha = \widetilde{\alpha}|_\mathcal{R}$ of $\widetilde{\alpha}$ to $\mathcal{R}$ factors through the quotient group $H = H_\mathrm{m}/(\mathbb{Z}z_0)$. Hence α is indeed an action of H. Since $\mathcal{R}_{0,1}$ is a factor of type II_∞ and

$$\begin{aligned}\tau\circ\widetilde{\alpha}_{z_0} &= \tau\circ\theta_{\mathrm{m}(z_0)} = e^{-\mathrm{m}(z_0)}\tau = e^{-T'}\tau \\ &= \lambda\tau,\end{aligned}$$

the fixed point subalgebra $\mathcal{R}$ is a factor of type III_λ and the pair $\{\mathcal{R}_{0,1}, \widetilde{\alpha}_{z_0}\}$ is the discrete core of the factor $\mathcal{R}$. Since $\mathcal{R}_{0,1}$ is AFD, $\mathcal{R}$ is approximately finite dimentsional by the grand theorem of Connes, [Cnn5]. As z_0 is a central element of H_m, $\widetilde{\alpha}(H_\mathrm{m})$ leaves $\mathcal{R}$ globally invariant and hence its restriction to $\mathcal{R}$ makes sense. The inner part $\widetilde{\alpha}(N)$, which is given by the projective representation $\{U(g) : g \in N\}$, leaves $\mathcal{R}$ globally invariant, i.e., $U(g), g \in N$, normalizes $\mathcal{R}$; thus we have the inclusion $U(N) \subset \widetilde{\mathcal{U}}_0(\mathcal{R})$. Hence $N = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$. As in (1.1), we have

$$\mathrm{mod}(\alpha_h) = \mathrm{m}(h), \quad h \in H.$$

If $g, g_1, g_2 \in N$ and $h \in H$, then

$$\lambda(g; h) = U^*(g)\widetilde{\alpha}_h(U(h^{-1}gh));$$
$$U(g_1)U(g_2) = \mu(g_1; g_2)U(g_1g_2);$$
$$\nu_\alpha(g) = \partial_{\widetilde{\alpha}_{z_0}}(U(g)) = U(g)^*\widetilde{\alpha}_{z_0}(U(g)) = \lambda(g; z_0).$$

Hence $\chi(\widetilde{\alpha}) = [\lambda, \mu] \in \Lambda(H_\mathrm{m}, L, M, \mathbb{T})$ as required. Finally viewing ν_α as a homomorphism of N into $\mathbb{R}/T\mathbb{Z}$, we get $\nu_\alpha \in \mathrm{Hom}_G(N, \mathbb{R}/T\mathbb{Z})$ as in the assertion of the theorem.

ii) The assertion follows from the construction of $\alpha^{\lambda,\mu}$. ♡

Actions and Outer Actions of Two Commuting Automorphisms on an AFD factor $\mathcal{R}$ of type III_λ: In this case, we have to consider the free abelian group $G = \mathbb{Z}^2$ of rank two and its extension $G_{\mathrm{m}} \cong \mathbb{Z}^3$ relative to a homomorphism $\mathrm{m} : G \mapsto \mathbb{R}/T'\mathbb{Z}$. We fix a subgroup N of G, which is going to represent the inverse image $\alpha^{-1}(\mathrm{Cnt}(\mathcal{M}_\lambda))$ of the extended modular automorphism group. We assume that N is in the diagonal form, i.e., with a free basis $\{a_1, a_2\}$ of G the subgroup N is of the form:

$$N = p_1\mathbb{Z}a_1 + p_2\mathbb{Z}a_2.$$

Of course, one can choose p_1 and p_2 in such a way that $0 \leq p_1 \leq p_2$ and p_1 divides p_2, but to go beyond the finite rank case, we don't assume that p_1 is a divisor of p_2, which might make a matter slightly more involved. In the case that $G = \mathbb{Z}^2$, we have $\mathrm{H}^3(G, \mathbb{T}) = \{1\}$, so every outer action of G comes from an action of G. Since $\mathrm{H}^2(G, \mathbb{T}) \cong \mathbb{T} \neq \{1\}$, the outer conjugacy class of an action is bigger than the cocycle conjugacy class. To go further, we recall the reduced modified HJR-exact sequence, [KtT3, Theorem 3.11 page 116]:

$$\mathrm{H}^2(G, \mathbb{T}) \xrightarrow{\mathrm{Res}_{Q_{\mathrm{m}}}} \Lambda(G_{\mathrm{m}}, N, \mathbb{T}) \xrightarrow{\delta_{Q_{\mathrm{m}}}} \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G, N, \mathbb{T}) \xrightarrow{\mathrm{Inf}_{Q_{\mathrm{m}}}} \mathrm{H}^3(G, \mathbb{T}) = \{1\},$$

where $Q_{\mathrm{m}} = G_{\mathrm{m}}/N$. Here since $\mathrm{H}^3(G, \mathbb{T}) = \{1\}$, we don't have to consider the resolution group H and its subgroup M. To identify the subgroup $N \subset G$ as a subgroup of G_{m}, we need a little care. First, set

$$\begin{gathered}
z_0 = (0, T') \in G_{\mathrm{m}},\\
z_1 = \left(a_1, \frac{q_1 T'}{p_1}\right) \in G_{\mathrm{m}}, \quad z_2 = \left(a_2, \frac{q_2 T'}{p_2}\right),\\
b_1 = (p_1 a_1, 0) = p_1 z_1 - q_1 z_0 \in G_{\mathrm{m}},\\
b_2 = (p_2 a_2, 0) = p_2 z_2 - q_2 z_0 \in G_{\mathrm{m}},\\
N = \mathbb{Z}b_1 + \mathbb{Z}b_2 \subset G_{\mathrm{m}} = \mathbb{Z}z_0 + \mathbb{Z}z_1 + \mathbb{Z}z_2,\\
Q_{\mathrm{m}} = G_{\mathrm{m}}/N.
\end{gathered} \tag{1.8}$$

This gives the following coordinate system in G_{m} and N:

$$\begin{gathered}
g = e_{1,N}(g)b_1 + e_{2,N}(g)b_2 \in N, \quad \text{i.e.,} \quad e_{i,N}(g) = \frac{e_i(g)}{p_i}, \quad i = 1, 2,\\
h = \tilde{e}_0(h)z_0 + \tilde{e}_1(h)z_1 + \tilde{e}_2(h)z_2 \in G_{\mathrm{m}}.
\end{gathered} \tag{1.9}$$

Theorem 1.6 (Invariant). *Define* Z *and* B *by the following*:

$$\left.\begin{aligned}
\mathrm{Z} &= \left\{\begin{matrix} b = \{b(i,j) : i = 1,2, j = 0,1,2\} \in \mathbb{R}^6 : \\ p_j b(i,j) - q_j b(i,0) \in \mathbb{Z}, i = 1,2,\ j = 1,2 \end{matrix}\right\},\\
\mathrm{B} &= \left\{ b \in \mathrm{Z} : \begin{matrix} b(i,0), b(i,i) \in \mathbb{Z}, i = 1,2, \\ p_2 b(1,2) + p_1 b(2,1) \in \gcd(p_1,p_2)\mathbb{Z} \end{matrix}\right\},
\end{aligned}\right\} \tag{1.10}$$

and to each $b \in \mathrm{Z}$ *associate a cochain* $(\lambda_b, \mu_b) \in \mathrm{Z}(G_\mathrm{m}, N, \mathbb{T})$ *by:*

$$\begin{aligned}\lambda_b(g; h) &= \exp\left(2\pi\mathrm{i}\left(\sum_{i=1,2;j=0,1,2} b(i,j)e_{i,N}(g)\tilde{e}_j(h)\right)\right),\\ \mu_b(g_1; g_2) &= 1, \quad g, g_1, g_2 \in N,\ h \in G_\mathrm{m}.\end{aligned} \tag{1.11}$$

Then the cochain (λ_b, μ_b) *is a characteristic cocycle* $(\lambda_b, 1) \in \mathrm{Z}(G_\mathrm{m}, N, \mathbb{T})$. *The modular obstruction cocycle* $(c_b, \nu_b) = \delta(\lambda_b, 1) \in \mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ *corresponding to* $(\lambda_b, 1)$ *takes the form:*

$$\begin{aligned}c_b(\dot{g}_1; \dot{g}_2; \dot{g}_3) &= \lambda_b(\mathfrak{n}_N(\dot{g}_2; \dot{g}_3); \mathfrak{s}(\dot{g}_3)), \quad \dot{g}_1, \dot{g}_2, \dot{g}_3 \in Q_\mathrm{m},\\ &= \exp\left(2\pi\mathrm{i}\left(\sum_{i=1,2;j=0,1,2} \frac{b(i,j)\eta_{p_i}\Big([e_i(\dot{g}_2)]_{p_i}; [e_i(\dot{g}_3)]_{p_i}\Big)\{\tilde{e}_j(\dot{g}_1)\}_{p_j}}{p_i}\right)\right),\\ \nu_b(g) &= \left[T\sum_{i=1,2} b(i,0)e_{i,N}(g)\right]_T \in \mathbb{R}/T\mathbb{Z}, \quad g \in N,\end{aligned} \tag{1.12}$$

where for the notations η_{p_i} *and* $\mathfrak{n}_N$ *we refer the definition in* §3*, in particular* (3.8) *and* (3.14)*, and furthermore*

$$\{\tilde{e}_0(\dot{g}_1)\}_{p_0} = \tilde{e}_0(\dot{g}_1) \in \mathbb{Z}, \quad \dot{g}_1 \in Q_\mathrm{m}.$$

The (i,j)*-component* $\mathrm{Z}(i,j)$ *and* $\mathrm{B}(i,j)$ *of* Z *and* B *give more precise informations about the cocycles:*

i) *For* $i = 1, 2$*, we have*

$$\begin{aligned}\mathrm{Z}_b(i,i) &= \big\{z = (x,u) \in \mathbb{R}^2 : p_i x - q_i u \in \mathbb{Z}\big\},\\ \mathrm{B}_b(i,i) &= \mathbb{Z} \oplus \mathbb{Z}.\end{aligned} \tag{1.13}$$

The bicharacter $\lambda_z^{i,i}$ *on* $N \times G_\mathrm{m}$ *determined by:*

$$\lambda_z^{i,i}(g; h) = \exp(2\pi\mathrm{i}(xe_{i,N}(g)\tilde{e}_i(h) + ue_{i,N}(g)\tilde{e}_0(h))), \tag{1.14}$$

for each pair $g \in N, h \in G_\mathrm{m}$*, gives a characteristic cocycle of* $\mathrm{Z}(G_\mathrm{m}, N, \mathbb{T})$. *It is a coboundary if and only if* z *is in* $\mathrm{B}_b(i,i)$. *The corresponding cohomology class* $[\lambda_z^{i,i}] \in \Lambda_b(i,i)$ *has the parameterization:*

$$\begin{aligned}\big[\lambda_z^{i,i}\big] \in \Lambda(i,i) \sim \Big(&[p_i x - q_i u]_{\gcd(p_i,q_i)}, [-v_i x + u_i u]_\mathbb{Z}\Big)\\ &\in \mathbb{Z}_{\gcd(p_i,q_i)} \oplus (\mathbb{R}/\mathbb{Z}),\end{aligned} \tag{1.15}$$

where the integers u_i, v_i *are determined by:*

$$p_i u_i - q_i v_i = \gcd(p_i, q_i), \quad i = 1, 2,$$

through the Euclid algorithm. The associated modular obstruction cohomology class $\left(\left[c_z^{i,i}, \nu_z^{i,i}\right]\right) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(i,i)$ *corresponds to the class*:

$$\left.\begin{aligned} \left([p_i x - q_i u]_{\gcd(p_i,q_i)}, [-v_i x + u_i u]_{\mathbb{Z}}\right) \in \mathbb{Z}_{\gcd(p_i,q_i)} \oplus (\mathbb{R}/\mathbb{Z}),\\ \nu_z^{i,i}(g) = \left[Tue_{i,N}(g)\right]_T \in \mathbb{R}/T\mathbb{Z}, \end{aligned}\right\} \quad i = 1, 2.$$

ii) *With* $(i,j) = (1,2)$,

$$\begin{aligned} \mathrm{Z}_b(i,j) &= \left\{(x,u,y,v) \in \mathbb{R}^4 : p_j x - q_j u \in \mathbb{Z}, p_i y - q_i v \in \mathbb{Z}\right\};\\ \mathrm{B}_b(i,j) &= \left\{(x,u,y,v) \in \mathrm{Z}_b(i,j) : \begin{matrix} p_j x + p_i y \in \gcd(p_i,p_j)\mathbb{Z},\\ u, v \in \mathbb{Z} \end{matrix}\right\}. \end{aligned} \tag{1.16}$$

For each element $z = (x,u,y,v) \in \mathrm{Z}_b(i,j)$, *the corresponding bicharacter* λ_z *on* $N \times G_{\mathrm{m}}$:

$$\begin{aligned} \lambda_z^{i,j}(g;h) &= \exp\left(2\pi \mathrm{i}\left(xe_{i,N}(g)\tilde{e}_j(h) + ye_{j,N}(g)\tilde{e}_i(h)\right)\right)\\ &\quad \times \exp\left(2\pi \mathrm{i}\left(ue_{i,N}(g)\tilde{e}_0(h) + ve_{j,N}(g)\tilde{e}_0(h)\right)\right), \end{aligned} \tag{1.17}$$

for each pair $g \in N, h \in G_{\mathrm{m}}$ *is a characteristic cocycle in* $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. *It is a coboundary if and only if* $z \in \mathrm{B}_b(i,j)$. *The cohomology class* $[\lambda_z^{i,j}] \in \Lambda_b(i,j)$ *of* λ_z *corresponds to the parameter class*:

$$\left.\begin{aligned} [z] &= \begin{pmatrix} [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}}\\ [y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}}\\ [-uw_{i,j} + vw_{j,i}]_{\mathbb{Z}} \end{pmatrix}\\ &\in \begin{pmatrix} \left(\dfrac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z} \end{pmatrix}, \end{aligned}\right\} \tag{1.18}$$

where the integers $D(i,j), \cdots, w_{i,j}$ *are those such that*:

$$\left.\begin{gathered} D(i,j) = \gcd(p_i,p_j,q_i,q_j),\\ D_{i,j} = \gcd(p_i,p_j), \quad E_{i,j} = \gcd(q_i,q_j),\\ r_{i,j} = \frac{p_i}{D_{i,j}}, \quad r_{j,i} = \frac{p_j}{D_{i,j}} \quad s_{i,j} = \frac{q_i}{E_{i,j}}, \quad s_{j,i} = \frac{q_j}{E_{i,j}},\\ m_{i,j} = \frac{D_{i,j}}{D(i,j)}, \quad n_{i,j} = \frac{E_{i,j}}{D(i,j)},\\ q_i w_{i,j} + q_j w_{j,i} = E_{i,j}, \quad x_{i,j}D_{i,j} + y_{i,j}E_{i,j} = D(i,j). \end{gathered}\right\} \tag{1.19}$$

The associated modular obstruction class $\left(\left[c_z^{i,j}\right], \nu_z^{i,j}\right) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(i,j)$ *corresponds to the pair of the classes*:

$$[z] \in \begin{pmatrix} \left(\dfrac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z} \end{pmatrix},$$

$$\nu_z^{i,j}(g) = \left[T\left(u\frac{e_1(g)}{p_1} + v\frac{e_2(g)}{p_2}\right)\right]_T \in \mathbb{R}/T\mathbb{Z}, \quad g \in N.$$

The proof of this special case is not particularly simpler than the general case, so that we will discuss later in the general free abelian group case, see Theorem 4.2.

§2 Asymmetrization.

Set the notations:

$$X = \mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}, \quad X_1 = X\backslash\{1\}.$$

The signature of a permutation σ is the sign of the product:

$$\mathrm{sign}(\sigma) = \mathrm{sign}\left\{\prod_{i<j}(\sigma(j)-\sigma(i))\right\}.$$

Let S be the cyclic permutation:

$$S = (2, 3, \cdots, n, n+1, 1) \in \Pi(X), \tag{2.1}$$

whose signature is given by:

$$\mathrm{sign}(S) = (-1)^n. \tag{2.2}$$

Each element $\sigma \in \Pi(X_1)$ is identified with the element of $\Pi(X)$ such that

$$\sigma = (1, \sigma(2), \sigma(3), \cdots, \sigma(n), \sigma(n+1)) \in \Pi(X).$$

This identification of an element of $\Pi(X_1)$ with the corresponding element of $\Pi(X)$ preserves the signature of σ. Then the total permutation group $\Pi(X)$ is the disjoint union of the translations $\{S^k\Pi(X_1) : 0 \leq k \leq n\}$, i.e.,

$$\Pi(X) = \bigcup_{k=0}^{n} S^k\Pi(X_1), \quad \text{disjoint union.} \tag{2.3}$$

Definition 2.1. The *asymmetrization* $\mathrm{AS}\xi$ of $\xi \in \mathrm{C}^n(G, A)$ is defined by the following:

$$(\mathrm{AS}\xi)(g_1, g_2, \cdots, g_n) = \sum_{\sigma\in\Pi(\mathbb{Z}_n)} \mathrm{sign}(\sigma)\xi(g_{\sigma(1)}, g_{\sigma(2)}, \cdots, g_{\sigma(n)}). \tag{2.4}$$

Define $\pi_k : G^{n+1} \mapsto G^n$ by the following:

$$\pi_k(g_1, g_2, \cdots, g_n, g_{n+1}) = \begin{cases} (g_2, g_3, \cdots, g_n, g_{n+1}), & k = 0; \\ (g_1, \cdots, g_{k-1}, g_kg_{k+1}, g_{k+2}, \cdots, g_{n+1}), & 1 \leq k \leq n; \\ (g_1, g_2, \cdots, g_n), & k = n+1. \end{cases} \tag{2.5}$$

The boundary operation $d \in \operatorname{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ is then given by

$$d = \sum_{k=0}^{n+1} (-1)^k \circ \pi_k, \tag{2.6}$$

and

$$\partial\xi = d^*\xi, \quad \xi \in \mathrm{C}^{n+1}(G, \mathbb{T}).$$

We view the asymmetrization AS also as an element of $\operatorname{End}(\mathbb{Z}(G^n))$ determined by:

$$\mathrm{AS}(g_1, g_2, \cdots, g_n) = \sum_{\sigma \in \Pi(\mathbb{Z}_n)} \operatorname{sign}(\sigma)\big(g_{\sigma(1)}, g_{\sigma(2)}, \cdots, g_{\sigma(n)}\big).$$

Lemma 2.2. *The asymmetrization and the boundary operation are related in the following way*:

$$\mathrm{AS}\circ d = 0 \quad \textit{in} \ \operatorname{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n)).$$

Proof. Define $Q \in \operatorname{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ and $R \in \operatorname{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ by:

$$Q = \sum_{\sigma \in \Pi(X_1)} \sum_{j=1}^{n+1} \Big(\operatorname{sign}\left(S^{j-1}\sigma\right) \pi_0 S^{j-1}\sigma + (-1)^{n+1} \operatorname{sign}\left(S^j\sigma\right) \pi_{n+1} S^j \sigma\Big),$$

and

$$Rg = \sum_{\sigma \in \Pi(X_1)} \sum_{j=1}^{n+1} \operatorname{sign}(S^j\sigma) \sum_{k=1}^{n} (-1)^k \pi_k S^j \sigma g, \quad g \in G^{n+1}.$$

So we have

$$\mathrm{AS}\circ d = Q + R.$$

We know

$$\begin{aligned}
&\pi_0 S^{j-1}\sigma = \pi_{n+1} S^j \sigma, \quad 1 \le j \le n;\\
&\operatorname{sign}(S^{j-1}\sigma)\pi_0 S^{j-1}\sigma + (-1)^{n+1}\operatorname{sign}(S^j\sigma)\pi_{n+1}S^j\sigma\\
&\qquad = (-1)^{n(j-1)}\operatorname{sign}(\sigma)\pi_0 S^{j-1}\sigma + (-1)^{n+1}(-1)^{nj}\operatorname{sign}(\sigma)\pi_{n+1}S^j\sigma\\
&\qquad = 0.
\end{aligned}$$

Thus we get

$$Q = 0.$$

We need the notation $\sigma_{k,k+1}$ for the flip of k and $k+1$:

$$\sigma_{k,k+1} = (1, 2, \cdots, k-2, k-1, k+1, k, k+2, k+3, \cdots, n+1) \in \Pi(X).$$

Then we get

$$\operatorname{sign}(\sigma_{k,k+1}\rho)\pi_k\sigma_{k,k+1}\rho g+\operatorname{sign}(\rho)\pi_k\rho g=0,\quad \rho\in\Pi(X),1\leq k\leq n.$$

Hence we come to the following:

$$\begin{aligned}R&=\sum_{\sigma\in\Pi(X_1)}\sum_{j=1}^{n+1}\operatorname{sign}(S^j\sigma)\sum_{k=1}^{n}(-1)^k\pi_kS^j\sigma\\&=\sum_{k=1}^{n}(-1)^k\sum_{j=1}^{n+1}\sum_{\sigma\in\Pi(X_1)}\operatorname{sign}(S^j\sigma)\pi_kS^j\sigma\\&=\sum_{k=1}^{n}(-1)^k\sum_{\rho\in\Pi(X)}\operatorname{sign}(\rho)\pi_k\rho\\&=\sum_{k=1}^{n}(-1)^k\sum_{\rho\in\Pi_0(X)}(\operatorname{sign}(\rho)\pi_k\rho+\operatorname{sign}(\sigma_{k,k+1}\rho)\pi_k\sigma_{k,k+1}\rho)\\&=0,\end{aligned}$$

where $\Pi_0(X)$ is the group of even permutations of X, i.e., the alternating group. Therefore we conclude

$$\mathrm{AS}\circ d=Q+R=0.$$

This completes the proof. ♡

Let $\mathcal{A}$ be a G-module with action α. We recall the dimension shifting theorem and the dimension shift map ∂. First we define a new G-module $\widetilde{\mathcal{A}}$ as follows:

i) Let $\operatorname{Map}(G,\mathcal{A})$ be the module $\mathcal{A}^G$ of all $\mathcal{A}$-valued functions on G with pointwise addition.
ii) View the group $\mathcal{A}$ as the submodule of $\operatorname{Map}(G,\mathcal{A})$ of constant $\mathcal{A}$-valued functions.
iii) The action α of G on $\mathcal{A}$ is extended to the enlarged additive group $\operatorname{Map}(G,\mathcal{A})$ by:

$$(\alpha_hf)(g)=\alpha_h(f(gh)),\quad f\in\operatorname{Map}(G,\mathcal{A}),\quad g,h\in G.$$

iv) Form the quotient G-module:

$$\widetilde{\mathcal{A}}=\operatorname{Map}(G,\mathcal{A})/\mathcal{A}.$$

Thus we obtain the following equivariant short exact sequence:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathrm{Map}(G,\mathcal{A}) \longrightarrow \widetilde{\mathcal{A}} \longrightarrow 0. \tag{2.7}$$

The short exact sequence (2.7) splits in the following way:

i) First, set

$$j(f)(g) = f(g) - f(e), \quad f \in \mathrm{Map}(G,\mathcal{A}), g \in G,$$

where $e \in G$ is the neutral element of G. Then the map j is a homomorphism of $\mathrm{Map}(G,\mathcal{A})$ onto the subgroup $\mathrm{Map}_0(G,\mathcal{A})$ of all $\mathcal{A}$-valued functions on G vanishing at e. Then we get

$$\mathrm{Ker}(j) = \mathcal{A} \subset \mathrm{Map}(G,\mathcal{A}),$$

so that the map j is viewed as a bijection from $\widetilde{\mathcal{A}}$ onto $\mathrm{Map}_0(G,\mathcal{A})$.

ii) The map j transforms the action $\widetilde{\alpha}$ of G on $\widetilde{\mathcal{A}}$ to the action, denoted by $\widetilde{\alpha}$ again, on $\mathrm{Map}_0(G,\mathcal{A})$ defined by:

$$(\widetilde{\alpha}_h f)(g) = \alpha_h(f(gh)) - \alpha_h(f(h)), \quad g, h \in G, \quad f \in \mathrm{Map}_0(G,\mathcal{A}).$$

With the map j, we will identify $\widetilde{\mathcal{A}}$ and $\mathrm{Map}_0(G,\mathcal{A})$. Thus we have a short exact sequence:

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathrm{Map}(G,\mathcal{A}) \underset{\underset{\mathfrak{s}}{\longleftarrow}}{\xrightarrow{\ j\ }} \widetilde{\mathcal{A}} = \mathrm{Map}_0(G,\mathcal{A}) \longrightarrow 0$$

Let $\mathfrak{s}$ denote the embedding of $\widetilde{\mathcal{A}} = \mathrm{Map}_0(G,\mathcal{A}) \hookrightarrow \mathrm{Map}(G,\mathcal{A})$, which is a right inverse of the map j. If $\tilde{u} \in \mathrm{Z}_\alpha^{n-1}\Big(G, \widetilde{\mathcal{A}}\Big)$, then

$$0 = \partial_G \tilde{u} = j\left(\tilde{\partial}_G \mathfrak{s}(\tilde{u})\right),$$

where $\tilde{\partial}_G$ means the coboundary operator in $\mathrm{C}_{\widetilde{\alpha}}^n(G, \mathrm{Map}(G,\mathcal{A}))$, so that we have $\partial_G \mathfrak{s}(\tilde{u}) \in \mathrm{Z}_\alpha^n(G,\mathcal{A})$. We denote the cohomology class $\left[\tilde{\partial}_G \mathfrak{s}(\tilde{u})\right] \in \mathrm{H}_\alpha^n(G,\mathcal{A})$ by $\partial[\tilde{u}]$ for each $[\tilde{u}] \in \mathrm{H}_{\widetilde{\alpha}}^{n-1}\Big(G, \widetilde{\mathcal{A}}\Big)$. It is known as the dimension shift theorem that the map ∂ is an isomorphism of $\mathrm{H}_{\widetilde{\alpha}}^{n-1}\Big(G, \widetilde{\mathcal{A}}\Big)$ onto $\mathrm{H}_\alpha^n(G,\mathcal{A})$.

Definition 2.3. Suppose that the group G admits a torsion free central element $z_0 \in G$. A cocycle $c \in \mathrm{Z}_\alpha^n(G,\mathcal{A})$ is said to be of the *standard form* (relative to the central element z_0) if

i) For each $k_1, \cdots k_n \in \mathbb{Z}$ and $g_1, g_2, \cdots, g_n \in G$,

$$c(z_0^{k_1} g_1, \cdots, z_0^{k_n} g_n) = \alpha_{g_1}\Big(d_c(k_1; g_2, \cdots, g_n)\Big) + c(g_1, g_2, \cdots, g_n); \tag{2.8}$$

ii) The map $k \in \mathbb{Z} \mapsto d_c(k; g_2, g_3, \cdots, g_n) \in \mathcal{A}$ is in $\mathrm{Z}^1_{\alpha_{z_0}}(\mathbb{Z}, \mathcal{A})$ for each $g_2, g_3, \cdots, g_n \in G$, i.e.,

$$\begin{aligned} &d_c(k+\ell; g_2, g_3, \cdots, g_n) \\ &\qquad = d_c(k; g_2, g_3, \cdots, g_n) + \alpha^k_{z_0}\Big(d_c(\ell; g_2, g_3, \cdots, g_n)\Big). \end{aligned} \tag{2.9}$$

iii) For each $k \in \mathbb{Z}$ and $g_1, g_2, \cdots, g_n \in G$, we have

$$\begin{aligned} &(\partial_G d_c)(k; g_1, g_2, \cdots, g_n) \\ &\qquad = \alpha^k_{z_0}\Big(c(g_1, g_2, \cdots, g_n)\Big) - c(g_1, g_2, \cdots, g_n). \end{aligned} \tag{2.10}$$

Remark 2.4. The cocycle identity (2.8) can be fulfilled automatically if d_c is chosen in such a way that

$$\begin{gathered} c\Big(z_0 g_1, z_0^{k_2} g_2, \cdots, z_0^{k_n} g_n\Big) = \alpha_{g_1}\Big(d_c(g_2, g_3, \cdots, g_n)\Big) + c(g_1, g_2, \cdots, g_n), \\ (\partial_G d_c)(g_1, g_2, \cdots, g_n) = \alpha_{z_0}\Big(c(g_1, g_2, \cdots, g_n)\Big) - c(g_1, g_2, \cdots, g_n). \end{gathered}$$

Because $d_c(k; g_2, g_3, \cdots, g_n)$ can be obtained inductively by:

$$\begin{aligned} &d_c(k; g_2, g_3, \cdots, g_n) \\ &\qquad = d_c(g_2, g_3, \cdots, g_n) + \alpha_{z_0}\Big(d_c(k-1; g_2, g_3, \cdots, g_n)\Big). \end{aligned} \tag{2.11}$$

In the sequel, we often write $d_c(g_2, g_3, \cdots, g_n)$ for the d-part of a standard cocycle c without referring to the first variable k in $d_c(k; g_2, g_3, \cdots, g_n)$.

Lemma 2.5. *In the above context, every cocycle $c \in \mathrm{Z}^n_\alpha(G, \mathcal{A})$ is cohomologous to a cocycle c_{s} of the standard form.*

Proof. For $n = 1$, the cocycle identity:

$$c(z_0^k g) = \alpha_g\big(c\big(z_0^k\big)\big) + c(g), \quad k \in \mathbb{Z}, g \in G,$$

shows that with $d_c(k) = c\big(z_0^k\big)$ the cochain d_c satisfies the condition (i). Now we have

$$\begin{aligned} \alpha^k_{z_0}\Big(c(g)\Big) - c(g) &= c\big(z_0^k g\big) - c\big(z_0^k\big) - c(g) \\ &= c(g) + \alpha_g\Big(c\big(z_0^k\big)\Big) - c\big(z_0^k\big) - c(g) \\ &= \alpha_g\Big(d_c(k)\Big) - d_c(k) \\ &= \Big(\partial_G d_c\Big)(k; g), \end{aligned}$$

which shows the property (ii) for c and d_c.

Now assume that our claim is valid for $1, \cdots, n-1$ and for any G-module $\{\mathcal{A}, \alpha\}$.

Choose an equivariant short exact sequence:

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } M \underset{\underset{\mathfrak{s}}{\longleftarrow}}{\xrightarrow{\ j\ }} \widetilde{\mathcal{A}} \longrightarrow 0$$

such that

i) $\mathrm{H}^n_\alpha(G, M) = \{0\}, n \geq 1$

ii) the cross-section $\mathfrak{s} : \widetilde{\mathcal{A}} \mapsto M$ is a homomorphism of $\widetilde{\mathcal{A}}$ into M, but not equivariant,

so that the map $\partial_G \mathfrak{s} : \mathrm{Z}^{n-1}_\alpha\left(G, \widetilde{\mathcal{A}}\right) \mapsto \mathrm{Z}^n_\alpha(G, \mathcal{A})$ gives rise to an isomorphism $\partial : \mathrm{H}^{n-1}_\alpha\left(G, \widetilde{\mathcal{A}}\right) \mapsto \mathrm{H}^n_\alpha(G, \mathcal{A})$. For a standard cocycle $\tilde{c} \in \mathrm{Z}^{n-1}_\alpha\left(G, \widetilde{\mathcal{A}}\right)$, we set, for each $z_0^{k_1} g_1, \cdots, z_0^{k_{n-1}} g_{n-1} \in G$,

$$\begin{aligned}\bar{c}\left(z_0^{k_1} g_1, \cdots, z_0^{k_{n-1}} g_{n-1}\right) = \alpha_{g_1}\Big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \cdots, g_{n-1}))\Big)& \\ + \mathfrak{s}(\tilde{c}(g_1, g_2, g_3, \cdots, g_{n-1}))&.\end{aligned}$$

Since $j(\bar{c}) = \tilde{c}$, we have

$$c = \partial_G \bar{c} \in \mathrm{Z}^n_\alpha(G, \mathcal{A}).$$

We then compute

$$\begin{aligned}
c&\left(z_0^{k_1} g_1, \cdots, z_0^{k_n} g_n\right) = (\partial_G \bar{c})\left(z_0^{k_1} g_1, \cdots, z_0^{k_n} g_n\right)\\
&= \alpha_{z_0^{k_1} g_1}\left(\bar{c}\left(z_0^{k_2} g_2, z_0^{k_3} g_3, \cdots, z_0^{k_n} g_n\right)\right)\\
&\qquad + \sum_{j=1}^{n-1} (-1)^j \bar{c}\left(z_0^{k_1} g_1, \cdots, z_0^{k_j} g_j z_0^{k_{j+1}} g_{j+1}, \cdots, g_n\right)\\
&\qquad\qquad + (-1)^n \bar{c}\left(z_0^{k_1} g_1, \cdots, z_0^{k_{n-1}} g_{n-1}\right)\\
&= \alpha_{z_0^{k_1} g_1}\left[\alpha_{g_2}\Big(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \cdots, g_n))\Big) + \bar{c}(g_2, g_3, \cdots, g_n)\right]\\
&\quad - \left[\alpha_{g_1 g_2}\Big(\mathfrak{s}(d_{\tilde{c}}(k_1 + k_2; g_3, \cdots, g_n))\Big) + \bar{c}(g_1 g_2, g_3, \cdots, g_n)\right]\\
&\quad + \sum_{j=2}^{n-1} (-1)^j \left[\alpha_{g_1}\Big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, \cdots, g_j g_{j+1}, \cdots, g_n))\Big)\right.\\
&\qquad\qquad \left. + \bar{c}(g_1, \cdots, g_j g_{j+1}, \cdots, g_n)\Big)\right]\\
&\quad + (-1)^n \alpha_{g_1}\Big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \cdots, g_{n-1}))\Big)\\
&\qquad\qquad + (-1)^n \bar{c}(g_1, g_2, g_3, \cdots, g_{n-1})\\
&= (\partial_G \bar{c})(g_1, g_2, \cdots, g_n) + \alpha_{z_0^{k_1} g_1}\Big(\alpha_{g_2}(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \cdots, g_n)))\Big)\\
&\quad + \alpha_{g_1}\left(\alpha_{z_0}^{k_1}\Big(\bar{c}(g_2, g_3, \cdots, g_n)\Big) - \bar{c}(g_2, g_3, \cdots, g_n)\right)\\
&\quad - \alpha_{g_1 g_2}\left[\mathfrak{s}(d_{\tilde{c}}(k_1; g_3, \cdots, g_n)) + \alpha_{z_0}^{k_1}(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \cdots, g_n)))\right]
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{j=2}^{n-1} (-1)^j \alpha_{g_1}(\mathfrak{s}(d_{\tilde c}(k_1; g_2, \cdots, g_j g_{j+1}, \cdots, g_n)))\\
&\qquad + (-1)^n \Big(\alpha_{g_1}(\mathfrak{s}(d_{\tilde c}(k_1; g_2, g_3, \cdots, g_{n-1})))\Big)\\
&= (\partial_G \bar c)(g_1, g_2, \cdots, g_n)\\
&\qquad\quad + \alpha_{g_1}\Big[\Big(\alpha_{z_0}^{k_1}\Big(\bar c(g_2, g_3, \cdots, g_n)\Big) - \bar c(g_2, g_3, \cdots, g_n)\Big)\\
&\qquad\qquad\qquad - \alpha_{g_2}\Big(\mathfrak{s}(d_{\tilde c}(k_1; g_3, \cdots, g_n))\Big)\\
&\qquad\quad + \sum_{j=2}^{n-1} (-1)^j \mathfrak{s}(d_{\tilde c}(k_1; g_2, \cdots, g_j g_{j+1}, \cdots, g_n))\\
&\qquad\qquad\qquad + (-1)^n (\mathfrak{s}(d_{\tilde c}(k_1; g_2, g_3, \cdots, g_{n-1})))\Big]\\
&= (\partial_G \bar c)(g_1, g_2, \cdots, g_n)\\
&\qquad\quad + \alpha_{g_1}\Big[\alpha_{z_0}^{k_1}\Big(\bar c(g_2, g_3, \cdots, g_n) - \bar c(g_2, g_3, \cdots, g_n)\Big)\\
&\qquad\qquad\qquad - \partial_G(\mathfrak{s}\circ d_{\tilde c})(k_1; g_2, g_3, \cdots, g_n)\Big].
\end{aligned}
$$

Consequently, we get

$$
c\Big(z_0^{k_1} g_1, \cdots, z_0^{k_n} g_n\Big) = \alpha_{g_1}(d_c(k_1; g_2, g_3, \cdots, g_n)) + c(g_1, g_2, \cdots, g_n)
$$

with

$$
\begin{aligned}
c(g_1, g_2, \cdots, g_n) &= (\partial_G \bar c)(g_1, g_2, \cdots, g_n),\\
d_c(g_2, g_3, \cdots, g_n) &= \alpha_{z_0}\Big(\bar c(g_2, g_3, \cdots, g_n)\Big) - \bar c(g_2, g_3, \cdots, g_n)\\
&\qquad\qquad - \partial_G(\mathfrak{s}\circ d_{\tilde c})(g_2, g_3, \cdots, g_n).
\end{aligned}
$$

We now check the requirement (2.10) for d_c and c:

$$
\begin{aligned}
&\alpha_{z_0}\Big(c(g_1, g_2, \cdots, g_n)\Big) - c(g_1, g_2, \cdots, g_n)\\
&\quad = \alpha_{z_0}\Big(\partial_G \bar c(g_1, g_2, \cdots, g_n)\Big) - \partial_G \bar c(g_1, g_2, \cdots, g_n)\\
&\quad = \partial_G\Big(\alpha_{z_0}(\bar c(g_1, g_2, \cdots, g_n)) - \bar c(g_1, g_2, \cdots, g_n)\Big)\\
&\quad = \partial_G\Big(d_c(g_2, g_3, \cdots, g_n) + \partial_G \mathfrak{s}\circ d_{\tilde c}(g_2, g_3, \cdots, g_n)\Big)\\
&\quad = \partial_G d_c(g_2, g_3, \cdots, g_n).
\end{aligned}
$$

Thus the cocycle c is standard. This completes the proof. ♡

We now state the main result on the asymmetrization which extends the work of Olesen-Pedersen -Takesaki, [OPT]:

Theorem 2.6. *Let Q be a countable torsion free abelian group.*

i) *The asymmetrization* AS *maps the group* $\mathrm{Z}^n(Q, \mathbb{T})$ *of* $\mathbb{T}$*-valued* n*-th cocycles onto the compact group* $X^n(Q, \mathbb{T})$ *of all asymmetric multi-characters on* n *variables of* Q*.*

ii) *The following sequence is exact for each* $n \in \mathbb{N}$:

$$1 \longrightarrow \mathrm{B}^n(Q,\mathbb{T}) \longrightarrow \mathrm{Z}^n(Q,\mathbb{T}) \xrightarrow{\mathrm{AS}} X^n(Q,\mathbb{T}) \longrightarrow 1.$$

Consequently,

$$\mathrm{H}^n(\mathbb{Z}^m,\mathbb{T}) \cong X^n(\mathbb{Z}^m,\mathbb{T}) \cong \begin{cases} \mathbb{T}^{\frac{m!}{n!(m-n)!}} & m \geq n \\ 0 & m < n \ . \end{cases}$$

More generally, if Q *is a countable torsion free abelian group, then the cohomology group* $\mathrm{H}^n(Q,\mathbb{T})$ *is naturally isomorphic to the Pontrjagin - Kampen dual of the* n*-th exterior power* $Q \wedge Q \wedge \cdots \wedge Q$ *of* Q:

$$\mathrm{H}^n(Q,\mathbb{T}) \cong \textit{The Pontrjagin - Kampen Dual of } (Q \wedge Q \wedge \cdots \wedge Q).$$

iii) *The group* $X^n(Q,\mathbb{T})$ *is a subgroup of* $\mathrm{Z}^n(Q,\mathbb{T})$ *such that*

$$\mathrm{Z}^n(Q,\mathbb{T}) = X^n(Q,\mathbb{T})\mathrm{B}^n(Q,\mathbb{T});$$
$$X^n(Q,\mathbb{T}) \cap \mathrm{B}^n(Q,\mathbb{T}) = \mathrm{Ker}(\text{Power } n!),$$

and

$$\mathrm{AS}c = c^{n!}, \quad c \in X^n(Q,\mathbb{T}).$$

Remark 2.7. If the group Q has torsion, then the theorem fails as seen in the case that $Q = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}, p \geq 2$, $\mathrm{H}^3(Q,\mathbb{T}) \cong \mathbb{Z}_p$ and $X^3(Q,\mathbb{T}) = \{0\}$.

For the proof, we need some preparation. First, if $n = 1$, then our assertion is trivially true for any abelian group Q with no assumption on torsion. We then assume that our assertion is true for cocycle dimension $1, \cdots, n-1$ with $n \in \mathbb{N}$ fixed and for any torsion free abelian group Q. With this induction hypothesis, we prepare a couple of lemmas for cocycle dimension n.

Lemma 2.8. i) *If* M *is an abelian group such that a cocycle* $c \in \mathrm{Z}^n(M,\mathbb{T})$ *is a coboundary if and only if* $\mathrm{AS}c = 1$*, then the same is true for the product group* $Q = M \times \mathbb{Z}$.

ii) *If* M *is an abelian group such that the asymmetrization* $\mathrm{AS}c$ *of each cocycle* $c \in \mathrm{Z}^n(M,\mathbb{T})$ *is a multi-character, then the same is true for the product group* $Q = M \times \mathbb{Z}$.

Proof. Let z_0 denote the distinguished element of Q associated with the product decomposition $Q = M \times \mathbb{Z}$ so that every element $q \in Q$ is written uniquely in the form $q = mz_0^k, m \in M, k \in \mathbb{Z}$.

i) The triviality of the asymmetrization of a coboundary was proven in Lemma 2.2. Thus we prove the converse. Suppose that $\mathrm{AS}c = 1, c \in \mathrm{Z}^n(Q,\mathbb{T})$. By Lemma 2.5 the cocycle c is cohomologous to a cocycle c_{s} of standard form and $\mathrm{AS}c_{\mathrm{s}} = \mathrm{AS}c = 1$ by Lemma 2.2. So we may and do assume that c is standard:

$$c(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n) = d_c(p_2,p_3,\cdots,p_n)^{\ell_1} c_M(p_1,p_2,\cdots,p_n)$$

where $\tilde{p}_i = p_i z_0^{\ell_i} \in Q = M \times \mathbb{Z}$. As Q does not act on $\mathbb{T}$, the d-part d_c is a cocycle in $\mathrm{Z}^{n-1}(Q, \mathbb{T})$.

We look at the asymmetrization of c:

$$
\begin{aligned}
(\mathrm{AS}c)(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) &= \prod_{\sigma \in S_n} \Big(d_c\big(p_{\sigma(2)}, p_{\sigma(3)}, \cdots, p_{\sigma(n)}\big)^{\ell_{\sigma(1)}} \\
&\qquad\qquad \times c_M\big(p_{\sigma(1)}, p_{\sigma(2)}, \cdots, p_{\sigma(n)}\big)\Big)^{\mathrm{sign}\ \sigma} \\
&= \prod_{\sigma \in S_n} d_c\big(p_{\sigma(2)}, p_{\sigma(3)}, \cdots, p_{\sigma(n)}\big)^{\ell_{\sigma(1)}\mathrm{sign}\ \sigma} \\
&\qquad\qquad \times \prod_{\sigma \in S_n} c_M\big(p_{\sigma(1)}, p_{\sigma(2)}, \cdots, p_{\sigma(n)}\big)^{\mathrm{sign}\ \sigma},
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(\mathrm{AS}c)(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) &= \prod_{\sigma \in S_n} d_c\big(p_{\sigma(2)}, p_{\sigma(3)}, \cdots, p_{\sigma(n)}\big)^{\ell_{\sigma(1)}\mathrm{sign}\ \sigma} \\
&\qquad\qquad \times (\mathrm{AS}c_M)(p_1, p_2, \cdots, p_n).
\end{aligned}
\tag{2.12}
$$

To compute the first term of the above expression, we take a closer look at the permutation group S_n. In particular, we have to pay attention to the fact that the first term in the variables of d_c is mission. To this end, we fix $k, 1 \le k \le n$, which represents the missing term in d_c, and consider the cyclic permutation:

$$
S_{n-1}(k) = (1, 2, \cdots, k-1, k+1, \cdots, n) \in \Pi\Big(\{1, 2, \cdots, k-1, k+1, \cdots, n\}\Big).
$$

For $\sigma = (k, \sigma(2), \sigma(3), \cdots, \sigma(n)) \in S_n$, define $\rho,\ \tilde{\rho}$ and $\tilde{\sigma}$ as follows:

$$
\begin{aligned}
\rho &= S^{(n-k+1)}\sigma \\
&= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n \\ \sigma(n-k+2) & \sigma(n-k+3) & \cdots & \sigma(n) & k & \sigma(2) & \cdots & \sigma(n-k+1) \end{pmatrix}; \\
\tilde{\rho} &= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k+1 & \cdots & n \\ \sigma(n-k+2) & \sigma(n-k+3) & \cdots & \sigma(n) & \sigma(2) & \cdots & \sigma(n-k+1) \end{pmatrix}; \\
\tilde{\sigma} &= S_{n-1}(k)^{k-1}\tilde{\rho} \\
&= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k+1 & \cdots & n \\ \sigma(2) & \sigma(3) & \cdots & \sigma(k) & \sigma(k+1) & \cdots & \sigma(n) \end{pmatrix} \\
&= \big(\sigma(2), \sigma(3), \cdots, \sigma(n)\big).
\end{aligned}
$$

Then observing sign $\tilde{\rho} =$ sign ρ, we compute

$$
\begin{aligned}
\mathrm{sign}\ \sigma &= \mathrm{sign}\ S^{k-1}\mathrm{sign}\ \rho = (-1)^{(n-1)(k-1)}\mathrm{sign}\ \tilde{\rho} \\
&= (-1)^{(n-1)(k-1)}\mathrm{sign}\big(S_{n-1}(k)^{n-k}\big)\mathrm{sign}\ \tilde{\sigma} \\
&= (-1)^{(n-1)(k-1)+(n-2)(n-k)}\mathrm{sign}\ \tilde{\sigma} = (-1)^{k-1}\mathrm{sign}\ \tilde{\sigma}.
\end{aligned}
$$

Hence the first term of (2.12) becomes the following:

$$\begin{aligned}
&\prod_{\sigma\in S_n}\left\{d_c(p_{\sigma(2)},p_{\sigma(3)},\cdots,p_{\sigma(n)})\right\}^{\ell_{\sigma(1)}\text{sign }\sigma}\\
&\qquad=\prod_{k=1}^{n}\left\{\prod_{\tilde\sigma\in S_{n-1}(k)}\left\{d_c(p_{\tilde\sigma(1)},p_{\tilde\sigma(2)},\cdots,p_{\tilde\sigma(n-1)})\right\}^{\text{sign }\tilde\sigma}\right\}^{\ell_k(-1)^{k-1}}\\
&\qquad=\prod_{k=1}^{n}\left\{(\mathrm{AS}d_c)(p_1,p_2,\cdots,\breve{p}_k,\cdots,p_n)\right\}^{\ell_k(-1)^{k-1}}
\end{aligned}$$

where the notation $\smile$ stands for removing the corresponding variable. Thus (2.12) is replaced by the following:

$$\begin{aligned}
&(\mathrm{AS}c)(\tilde p_1,\tilde p_2,\cdots,\tilde p_n)\\
&\qquad=\prod_{k=1}^{n}\left\{(\mathrm{AS}d_c)(p_1,p_2,\cdots,\breve{p}_k,\cdots,p_n)\right\}^{\ell_k(-1)^{k-1}}\\
&\qquad\qquad\qquad\qquad\times(\mathrm{AS}c_M)(p_1,p_2,\cdots,p_n).
\end{aligned}\tag{2.12$'$}$$

The condition $\mathrm{AS}c=1$ yields that:

$$\begin{gathered}
\mathrm{AS}c_M=1\quad\text{with } \ell_k=0,k=1,\dots,n;\\
\mathrm{AS}d_c=1\quad\text{with } \ell_1=1,\ell_k=0,k=2,\dots,n,\ p_1=e.
\end{gathered}$$

Hence c_M and d_c are both coboundaries by the induction hypothesis. Choose $b\in\mathrm{C}^{n-1}(M,\mathbb{T})$ and $a\in\mathrm{C}^{n-2}(M,\mathbb{T})$ such that

$$c_M=\partial_{M}b\quad\text{and}\quad d_c=\partial_{M}a.$$

Then the cocycle c has the form:

$$\begin{aligned}
c(\tilde p_1,\tilde p_2,\cdots,\tilde p_n)&=d_c(p_2,p_3,\cdots,p_n)^{\ell_1}c(p_1,p_2,\cdots,p_n)\\
&=((\partial_{M}a)(p_2,p_3,\cdots,p_n))^{\ell_1}(\partial_{M}b)(p_1,p_2,\cdots,p_n).
\end{aligned}$$

Setting

$$f(\tilde p_1,\tilde p_2,\cdots,\tilde p_{n-1})=a(p_2,p_3,\cdots,p_{n-1})^{-\ell_1}b(p_1,p_2,\cdots,p_{n-1}),$$

for $\tilde p_i=z_0^{\ell_i}p_i\in Q, i=1,\cdots,n-1$, we compute

$$\begin{aligned}
&(\partial_{Q}f)(\tilde p_1,\tilde p_2,\cdots,\tilde p_n)\\
&\qquad=f(\tilde p_2,\tilde p_3,\cdots,\tilde p_n)\times\prod_{k=1}^{n-1}f(\tilde p_1,\cdots,\tilde p_k\tilde p_{k+1},\cdots,\tilde p_n)^{(-1)^k}\\
&\qquad\qquad\qquad\qquad\times f(\tilde p_1,\tilde p_2,\cdots,\tilde p_{n-1})^{(-1)^n}
\end{aligned}$$

$$= a(p_3, \cdots, p_n)^{-\ell_2} a(p_3, \cdots, p_n)^{\ell_1+\ell_2} \times \prod_{k=2}^{n-1} a(p_2, \cdots, p_k p_{k+1}, \cdots, p_n)^{-\ell_1(-1)^k} \times a(p_2, p_3, \cdots, p_{n-1})^{-\ell_1(-1)^n}$$
$$\times b(p_2, p_3, \cdots, p_n) \times \prod_{k=1}^{n-1} b(p_1, \cdots, p_k p_{k+1}, \cdots, p_n)^{(-1)^k} \times b(p_1, p_3, \cdots, p_n)^{(-1)^n}$$

$$\begin{aligned}
&= a(p_3, \cdots, p_n)^{\ell_1} \prod_{k=2}^{n-1} a(p_2, \cdots, p_k p_{k+1}, \cdots, p_n)^{-\ell_1(-1)^k} \\
&\qquad \times a(p_2, p_3, \cdots, p_{n-1})^{-\ell_1(-1)^n} \times (\partial_M b)(p_1, p_2, \cdots, p_n) \\
&= ((\partial_M a)(p_2, p_3, \cdots, p_n))^{\ell_1} (\partial_M b)(p_1, p_2, \cdots, p_n) \\
&= c(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n).
\end{aligned}$$

Therefore c is a coboundary. This completes the proof of the assertion (i).

ii) Fix a standard cocycle $c \in \mathrm{Z}^n(Q, \mathbb{T})$:

$$c(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = d_c(p_2, p_3, \cdots, p_n)^{\ell_1} c(p_1, p_2, \cdots, p_n)$$

with $d_c \in \mathrm{Z}^{n-1}(M, \mathbb{T})$ and $c_M \in \mathrm{Z}^n(M, \mathbb{T})$. Observing that $\mathrm{AS}c_M$ and $\mathrm{AS}d_c$ are both multi-characters by the assumptions, we compute with (2.13), for $\tilde{q}_1 = q_1 z_0^{k_1}$,

$$\begin{aligned}
&(\mathrm{AS}c)(\tilde{p}_1\tilde{q}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \\
&\quad = (\mathrm{AS}d_c)(p_2, \cdots, p_n)^{\ell_1+k_1} \\
&\qquad \times \prod_{j=2}^{n} \left\{ (\mathrm{AS}d_c)(p_1q_1, p_2, \cdots, \breve{p}_j, \cdots, p_n) \right\}^{\ell_j(-1)^{j-1}} \\
&\qquad\quad \times (\mathrm{AS}c_M)(p_1q_1, p_2, \cdots, p_n) \\
&\quad = (\mathrm{AS}d_c)(p_2, \cdots, p_n)^{\ell_1} \\
&\qquad \times \prod_{j=2}^{n} \left\{ (\mathrm{AS}d_c)(p_1, p_2, \cdots, \breve{p}_j, \cdots, p_n) \right\}^{\ell_j(-1)^{j-1}} \\
&\qquad \times (\mathrm{AS}d_c)(p_2, \cdots, p_n)^{k_1} \\
&\qquad\quad \times \prod_{j=2}^{n} \left\{ (\mathrm{AS}d_c)(q_1, p_2, \cdots, \breve{p}_j, \cdots, p_n) \right\}^{\ell_j(-1)^{j-1}} \\
&\qquad\qquad \times (\mathrm{AS}c_M)(p_1, p_2, \cdots, p_n)(\mathrm{AS}c_M)(q_1, p_2, \cdots, p_n) \\
&\quad = (\mathrm{AS}c)(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n)(\mathrm{AS}c)(\tilde{q}_1, \tilde{p}_2, \cdots, \tilde{p}_n).
\end{aligned}$$

Thus ASc is indeed multiplicative on the first variable, so that it is an asymmetric multi-character of $Q = M \times \mathbb{Z}$. ♡

Lemma 2.9. *Suppose that $c \in \mathrm{Z}^n(Q, \mathbb{T})$ has a trivial asymmetrization, i.e.,* $\mathrm{AS}c = 1$. *Assume the following*:

a) *M is a finitely generated subgroup of Q;*
b) *$a_0 \in Q$ but not in M;*
c) *$f \in \mathrm{C}^{n-1}(M, \mathbb{T})$ cobounds the restriction c_M of c to M, i.e.,*

$$\partial_M f = c_M.$$

Then the cochain f has an extension to the subgroup $N = \langle M, a_0\rangle$ generated by M and a_0 such that

$$\partial_N f = c_N,$$

where c_N is the restriction of c to the subgroup N.

Proof. To apply the structure theory of abelian groups, we use the additive group operation in the group Q. From the general theory of abelian groups, it follows that M and N are both free abelian groups and there exists a free basis $\{z_1, z_2, \cdots, z_m\}$ of N and non-negative integers $\{p_1, p_2, \cdots, p_r\} \subset \mathbb{Z}_+, 1 \le r \le m$, such that

$$N = \langle z_1, z_2, \cdots, z_m\rangle, \quad M = \langle p_1 z_1, \cdots, p_r z_r\rangle.$$

With the assumption for $n-1$, every $(n-1)$-cocycle $\mu \in \mathrm{Z}^{n-1}(M, \mathbb{T})$ is cohomologous to an asymmetric multi-character μ_{a}, i.e., there exist $a_{i_1, i_2, \cdots, i_r} \in \mathbb{R}$ such that

$$\begin{aligned}\mu_{\mathrm{a}}&(g_1, g_2, \cdots, g_{n-1})\\ &= \exp\Bigg[2\pi\mathrm{i} \sum_{\substack{i_j \in \{1,2,\cdots,r\}\\ 1\le i_1<i_2<\cdots<i_{n-1}\le n-1}} a_{i_1,i_2,\cdots,i_{n-1}}\\ &\qquad \times \Big(e_{i_1,M} \wedge e_{i_2,M} \wedge \cdots \wedge e_{i_{n-1},M}\Big)(g_1, g_2, \cdots, g_{n-1})\Bigg]\end{aligned}$$

where $\{e_{i,M} : 1 \le i \le r\}$ is the coordinate sysem of M relative to the basis $\{p_1 z_1, \cdots, p_r z_r\}$. Setting

$$\begin{aligned}\nu_a&(g_1, g_2, \cdots, g_{n-1})\\ &= \exp\Bigg\{2\pi\mathrm{i} \sum_{\substack{i_j \in \{1,2,\cdots,r\}\\ 1\le i_1<i_2<\cdots<i_{n-1}\le n-1}} \frac{a_{i_1,i_2,\cdots,i_{n-1}}}{p_{i_1} p_{i_2} \cdots p_{i_{n-1}}}\\ &\qquad\qquad \times \Big(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{n-1}}\Big)(g_1, g_2, \cdots, g_{n-1})\Bigg\}\end{aligned}$$

where $\{e_i : 1 \le i \le m\}$ is the coordinate system of N relative to the basis $\{z_1, \cdots, z_m\}$, we obtain an extension ν of μ_{a}. Choose $\xi \in \mathrm{C}^{n-2}(M, \mathbb{T})$ so that $\mu = (\partial_M \xi)\mu_{\mathrm{a}}$ and extends ξ to a cochain $\xi \in \mathrm{C}^{n-2}(N, \mathbb{T})$. Then

the second cocycle $(\partial_N \xi)\nu$ gives an extension of the original $(n-1)$-cocycle $\mu \in \mathrm{Z}^{n-1}(M, \mathbb{T})$. Thus we obtain the surjectivity of the restriction map res : $\mu \in \mathrm{Z}^{n-1}(N, \mathbb{T}) \mapsto \mu_M \in \mathrm{Z}^{n-1}(M, \mathbb{T})$, i.e. the exactness of the sequence:

$$\mathrm{Z}^{n-1}(N, \mathbb{T}) \xrightarrow{\text{res}} \mathrm{Z}^{n-1}(M, \mathbb{T}) \longrightarrow 1.$$

By induction on generators, Lemma 2.9 yields that the restriction c_N of c to N is a coboundary. Hence there exists $\xi \in \mathrm{C}^{n-1}(N, \mathbb{T})$ such that $c_N = \partial_N \xi$. Then we have $\partial_M f = c_M = \partial_M \xi_M$, so that we obtain $\mu_M = \xi_M^{-1} f \in \mathrm{Z}^{n-1}(M, \mathbb{T})$. By the first arguments, we can extend μ_M to an element $\nu \in \mathrm{Z}^{n-1}(N, \mathbb{T})$. Set

$$f = \nu\xi \in \mathrm{C}^{n-1}(N, \mathbb{T}),$$

and the newly defined cochain f on N extends the original $f \in \mathrm{C}^{n-1}(M, \mathbb{T})$ and cobounds the cocycle c_N:

$$\partial_N f = (\partial_N \nu)(\partial_N \xi) = \partial_N \xi = c_N.$$

This completes the proof. ♡

We are now ready to complete the proof of Theorem 2.6, proceeding from cocycle dimension $1, \cdots, n-1$ to the cocycle dimension n.

Proof of Theorem 2.6. Suppose that $c \in \mathrm{Z}^n(Q, \mathbb{T})$ and $\mathrm{AS}c = 1$. Let $\{z_k : k \in \mathbb{N}\}$ be a sequence of generators of Q and let

$$M_m = \langle z_1, z_2, \cdots, z_m \rangle, \quad m \in \mathbb{N}.$$

The sequence $\{M_m\}$ is then increasing and $Q = \bigcup M_m$. The triviality assumption $\mathrm{AS}c = 1$ and Lemma 2.8 (i) yield that the restriction c_m of the cocycle c to each M_m is a coboundary, so that there exists $f_m \in \mathrm{C}^{n-1}(M_m, \mathbb{T})$ such that

$$c_m = \partial_{M_m} f_m.$$

The last lemma however allows us to choose the sequence $\{f_m\}$ in such a way that each f_m is an extension of the previous f_{m-1}. Hence the sequence $\{f_m\}$ gives a cochain $f \in \mathrm{C}^{n-1}(Q, \mathbb{T})$ such that $f|_{M_m} = f_m, m \in \mathbb{N}$, and therefore

$$\partial_Q f = c.$$

Thus we conclude that $\mathrm{Ker(AS)} \subset \mathrm{B}^n(Q, \mathbb{T})$. The inclusion, $\mathrm{Ker(AS)} \supset \mathrm{B}^n(Q, \mathbb{T})$, was proven in Lemma 2.2. Hence $\mathrm{Ker(AS)} = \mathrm{B}^n(Q, \mathbb{T})$.

Lemma 2.8 (ii) for $\{M_m\}_{m \in \mathbb{N}}$ yields that the asymmetrization $\mathrm{AS}c$ of every $c \in \mathrm{Z}^n(Q, \mathbb{T})$ is a multi-character.

Set $c_{\mathrm{a}} = \mathrm{AS}c$ for an arbitrary cocycle $c \in \mathrm{Z}^n(Q, \mathbb{T})$. Then $c_{\mathrm{a}} \in X^n(Q, \mathbb{T})$. Since Q is torsion free, the group $X^n(Q, \mathbb{T})$ is indefinitely divisible. So the $n!$-th power mapping: $\xi \in X^n(Q, \mathbb{T}) \mapsto \xi^{n!} \in X^n(Q, \mathbb{T})$ is surjective. But the asymmetrization AS on $X^n(Q, \mathbb{T})$ is precisely the $n!$-th power. Hence there

exists $\xi \in X^n(Q,\mathbb{T})$ such that $\mathrm{AS}\xi = \xi^{n!} = c_{\mathrm{a}}$. Now we have $\mathrm{AS}(\xi^{-1}c) = \xi^{-n!}c_{\mathrm{a}} = 1$. Thus $\xi^{-1}c \in \mathrm{B}^n(Q,\mathbb{T})$. Consequently, we conclude

$$\begin{aligned}\mathrm{Z}^n(Q,\mathbb{T}) &= X^n(Q,\mathbb{T})\mathrm{B}^n(Q,\mathbb{T});\\ X^n(Q,\mathbb{T})\cap \mathrm{B}^n(Q,\mathbb{T}) &= X^n(Q,\mathbb{T})\cap \mathrm{Ker(AS)}\\ &= \{c \in X^n(Q,\mathbb{T}) : c^{n!} = 1\}.\end{aligned}$$

This completes the proof. ♡

Corollary 2.10. *If G is a discrete abelian group, then the asymmetrization of every n-cocycle $c \in \mathrm{Z}^n(G,\mathbb{T})$ is a multi-character, i.e.,* $\mathrm{AS}c \in X^n(G,\mathbb{T})$.

Proof. Let F be a large enough free abelian group so that there exists a surjective homomorphism $\pi : F \mapsto G$. Consider the pullback $\pi^*(c)$ and its asymmetrization, $\mathrm{AS}\pi^*(c) = \pi^*(\mathrm{AS}c)$. It follows from Theorem 2.6 that the pull back $\pi^*(\mathrm{AS}c)$ is a multi-character of F, consequently the original asymmetrization $\mathrm{AS}c$ is a multi-character of G. ♡

§3. Universal Resolution for a Countable Discrete Abelian Group.

We discuss a universal resolution group for a countable discrete abelian group. We consider only the case that the abelian group under consideration has an infinitely many generators since the finitely generated case can be covered by the infinite generator case. Let $G = \mathbb{Z}^{<\mathbb{N}}$ be the free abelian group of a finite sequences of integers, i.e., every element $g \in G$ is of the form:

$$g = (g_1, g_2, \cdots, g_i, \cdots, g_\ell, 0, 0, \cdots), \quad g_i \in \mathbb{Z},$$

with $\ell = \ell(g) \in \mathbb{N}$, the last non-zero term of $g \in \mathbb{Z}^{<\mathbb{N}}$. With

$$a_i = (0, 0, \cdots, 0, \overset{\overset{i}{\downarrow}}{1}, 0, 0 \cdots), \tag{3.1}$$

every element $g \in \mathbb{Z}^{<\mathbb{N}}$ is written uniquely

$$g = \sum_{i\in\mathbb{N}} e_i(g)a_i. \tag{3.2}$$

We call $\{a_i : i \in \mathbb{N}\}$ the *standard basis* of $\mathbb{Z}^{<\mathbb{N}}$. We also fix a subgroup N of G which is generated by a sequence $\{p_i a_i : i \in \mathbb{N}\}$ with $p_i \in \mathbb{Z}_+, i \in \mathbb{N}$. We will use the matrix:

$$P = \begin{pmatrix} p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & \cdots \\ 0 & 0 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad N = P\mathbb{Z}^{<\mathbb{N}}.$$

Let M be the additive group of integer coefficient upper triangular matrices:

$$M = \left\{ m = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & \cdots \\ 0 & 0 & m_{23} & m_{24} & \cdots \\ 0 & 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} : m_{i,j} \in \mathbb{Z} \right\}$$

and set

$$e_{j,k}(m) = m_{jk}, \quad j < k, \ m \in M.$$

Let $a_i \wedge a_j, i < j$, be the element of M such that

$$e_{k,\ell}(a_i \wedge a_j) = \delta_{ik}\delta_{j\ell},$$

i.e., the matrix with only (i,j)-component 1 and all others 0, equivalently $a_i \wedge a_j, i < j$, is the (i,j)-matrix unit of M. Let $\mathfrak{n}_M$ be the M-valued second cocycle of G defined by:

$$e_{j,k}(\mathfrak{n}_M(g;h)) = e_j(g)e_k(h), \qquad g, h \in G, \ 1 \le j < k;$$

$$\mathfrak{n}_M(g;h) = \begin{pmatrix} 0 & e_1(g)e_2(h) & e_1(g)e_3(h) & e_1(g)e_4(h) & \cdots \\ 0 & 0 & e_2(g)e_3(h) & e_2(g)e_4(h) & \cdots \\ 0 & 0 & \ddots & e_3(g)e_4(h) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \end{pmatrix}. \tag{3.3}$$

Let H be the group extension of G associated with $\mathfrak{n}_M \in \mathrm{Z}^2(G, M)$:

$$H = M \times_{\mathfrak{n}_M} G \quad \text{and} \quad L = M \times_{\mathfrak{n}_M} N.$$

The group operation in H is given by:

$$(m,g)(n,h) = (m + n + \mathfrak{n}_M(g;h), g + h), \quad (m,g),(n,h) \in H.$$

The inverse $(m,g)^{-1}$ is given by:

$$(m,g)^{-1} = (-m + \mathfrak{n}_M(g,-g), -g)$$

because

$$\begin{gathered} (0,0) = (m,g)(m',g') = (m + m' + \mathfrak{n}_M(g;g'), g + g') \\ g' = -g, \quad m' = -m + \mathfrak{n}_M(g;g). \end{gathered}$$

To determine the commutater subgroup $[H,H]$, we compute the commutator:

$$\begin{aligned} &(m,g)(n,h)(m,g)^{-1}(n,h)^{-1}, \qquad\qquad (m,g),(n,h) \in H, \\ &\quad = (m,g)(n,h)(-m + \mathfrak{n}_M(g;g), -g)(-n + \mathfrak{n}_M(h;h); -h) \\ &\quad = (m + n + \mathfrak{n}_M(g,h), g + h) \\ &\qquad\qquad \times (-m - n + \mathfrak{n}_M(g;g) + \mathfrak{n}_M(h;h) + \mathfrak{n}_M(g;h), -g - h) \\ &\quad = (\mathfrak{n}_M(g;h) + \mathfrak{n}_M(g;g) + \mathfrak{n}_M(h;h) + \mathfrak{n}_M(g;h) + \mathfrak{n}_M(g+h; -(g+h)), 0) \\ &\quad = (\mathfrak{n}_M(g;h) - \mathfrak{n}_M(h;g), 0) \\ &\quad = \left(\sum_{j<k} (e_j(g)e_k(h) - e_j(h)e_k(g))(a_j \wedge a_k), 0 \right). \end{aligned}$$

This shows immediately the following:

Lemma 3.1. *The commutator subgroup $[H, H]$ of H is the center M of H.*

Proof. From the computation above, it follows that for each pair $j < k$

$$\mathfrak{s}_H(a_j)\mathfrak{s}_H(a_k)\mathfrak{s}_H(a_j)^{-1}\mathfrak{s}_H(a_k)^{-1} = a_j \wedge a_k,$$

with $\mathfrak{s}_H$ the cross-section of $\pi_0 : (m, g) \in H \mapsto g \in G$ given by

$$\mathfrak{s}_H(g) = (0, g) \in H, \quad g \in G,$$

so that the commutator subgroup $[H, H]$ contains the generators $a_j \wedge a_k, j < k$, of M. Thus our assertion follows. ♡

Theorem 3.2. *The pair $\{H, \pi_0\}$ is a universal resolution of the third cocycle group $\mathrm{Z}^3(G, \mathbb{T})$ of G. Consequently, if K is a countable discrete abelian group, then for any surjective homomorphism $\pi\colon \mathbb{Z}^{<\mathbb{N}} \mapsto K$, the composed map $\pi_K = \pi\circ\pi_0\colon H \mapsto K$ makes the pair $\{H, \pi_K\}$ a universal resolution of the third cocycle group $\mathrm{Z}^3(K, \mathbb{T})$.*

Proof. Since $\mathbb{Z}^{<\mathbb{N}}$ is a free abelian group on countably infinite generators, there exists a surjective homomorphism from G to any countable abelian group G. So it is sufficient to prove that

$$\pi_0^*\big(\mathrm{Z}^3(G, \mathbb{T})\big) \subset \mathrm{B}^3(H, \mathbb{T}).$$

For each triplet $\xi, \eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})$, we define a multi-homomorphism, called the *tensor product* and denoted by $\xi\otimes\eta\otimes\zeta \in \mathrm{C}^3(G, \mathbb{R})$, as follows:

$$(\xi\otimes\eta\otimes\zeta)(g; h; k) = \xi(g)\eta(h)\zeta(k), \quad g, h, k \in G.$$

Then the tensor product $\xi\otimes\eta\otimes\zeta$ generate the third cocycle group $\mathrm{Z}^3(G, \mathbb{R})$ up to coboundary, i.e.,

$$\Big\langle\Big\{\xi\otimes\eta\otimes\zeta : \xi, \eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})\Big\}\Big\rangle + \mathrm{B}^3(G, \mathbb{R}) = \mathrm{Z}^3(G, \mathbb{R}).$$

Now for each pair $\eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})$ we define a cochain $B_{\eta,\zeta} \in \mathrm{C}^1(H, \mathbb{R})$

$$B_{\eta,\zeta}(g) = \sum_{j<k} \eta(a_j)\zeta(a_k)e_{j,k}(m_0(g)), \quad g = (m_0(g), \pi_0(g)) \in H. \qquad (3.4)$$

Then we have

$$\begin{aligned}
\Big(\partial_H(\pi_0^*\xi\otimes B_{\eta,\zeta})\Big)(g_1;g_2;g_3) &= \xi(\pi_0(g_2))B_{\eta,\zeta}(g_3) - \xi(\pi_0(g_1)+\pi_0(g_2))B_{\eta,\zeta}(g_3)\\
&\quad + \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2g_3) - \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2)\\
&= -\xi(\pi_0(g_1))B_{\eta,\zeta}(g_3) + \xi(\pi_0(g_1))\left(\sum_{j<k}\eta(a_j)\zeta(a_k)e_{j,k}(m_0(g_2g_3))\right)\\
&\quad - \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2)\\
&= -\xi(\pi_0(g_1))B_{\eta,\zeta}(g_3)\\
&\quad + \xi(\pi_0(g_1))\left(\sum_{j<k}\eta(a_j)\zeta(a_k)\big(e_{j,k}(m_0(g_2)+m_0(g_3)+\pi_0^*\mathfrak{n}_M(g_2;g_3))\right)\\
&\quad - \xi(\pi_0(g_1)B_{\eta,\zeta}(g_2)\\
&= \xi(\pi_0(g_1))\left(\sum_{j<k}\eta(a_j)\zeta(a_k)e_j(\pi_0(g_2))e_k(\pi_0(g_3))\right).
\end{aligned}$$

Choosing $\xi,\eta,\zeta\in\mathrm{Hom}(G,\mathbb{T})$ to be $\xi=e_i,\eta=e_j$ and $\zeta=e_k$ for $i<j<k$, we obtain

$$\pi_0^*(e_i\otimes e_j\otimes e_k) = \partial_H\big(\pi_0^*e_i\otimes B_{e_j,e_k}\big).$$

Every third cocycle in $\mathrm{Z}^3(G,\mathbb{T})$ is cohomologous to a cocycle $c_a\in\mathrm{Z}^3(G,\mathbb{T})$ of the form:

$$c_a(g_1;g_2;g_3) = \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k}a(i,j,k)e_i(g_1)e_j(g_2)e_k(g_3)\right)\right). \tag{3.5}$$

So with $b_a\in\mathrm{C}^2(H,\mathbb{T})$ defined by:

$$b_a(g_1;g_2) = \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k}a(i,j,k)e_i(\pi_0(g_1))B_{e_j,e_k}(g_3)\right)\right), \tag{3.6}$$

we have

$$\pi_0^*c_a = \partial_H b_a. \tag{3.7}$$

Hence we get

$$\pi_0^*\big(\mathrm{Z}^3(G,\mathbb{T})\big)\subset\mathrm{B}^3(H,\mathbb{T}),$$

which concludes that the pair $\{H,\pi_0\}$ is a universal resolution of $\mathrm{Z}^3(G,\mathbb{T})$ and completes the proof. ♡

Corollary 3.3. *The μ-part of every characteristic cocycle $(\lambda,\mu)\in\mathrm{Z}(H,M,\mathbb{T})$ is trivial.*

Proof. Since $M\triangleleft H$ is central, λ is a bicharacter of $M\times H$, in particular $\lambda(m,\cdot)$ is a character of H for every $m\in M$. Hence it must vanish on

the commutator subgroup, i.e., $\lambda(m,n)=1$ for ever $m,n\in M$. Thus $\mu\in \mathrm{Z}^2(M,\mathbb{T})$ is a coboundary. ♡

Consider $(\lambda,\mu)\in \mathrm{Z}(H,L,\mathbb{T})$ with $L=M\times_{\mathfrak{n}_M}N$. We may and do assume the triviality $\mu_M=1$ of the restriction of μ to M. We then have the corresponding crossed extension:

$$1 \longrightarrow \mathbb{T} \longrightarrow E \underset{\longleftarrow_u}{\overset{j}{\longrightarrow}} L \longrightarrow 1$$

The triviality of μ_M means that the cross-section u is multiplicative on M, i.e., $u(mn)=u(m)u(n), m,n\in M$. Here we use the multiplicative group operation as M sits in the noncommutative group H.

Lemma 3.4. *If $\mathfrak{s}_H$ is a cross-section of the quotient map $\pi_0: H\mapsto \mathbb{Z}^{<\mathbb{N}}=H/M$ with $\mathfrak{n}_M=\partial\mathfrak{s}_H\in \mathrm{Z}^2(\mathbb{Z}^{<\mathbb{N}},M)$, then each characteristic cocycle in $\mathrm{Z}(H,L,M,\mathbb{T})$ is cohomologous to the one $(\lambda,\mu)\in \mathrm{Z}(H,L,M,\mathbb{T})$ such that:*

$$\lambda(m;n\mathfrak{s}_H(h))=\lambda(m;\mathfrak{s}_H(h)),\quad m,n\in M,\ h\in\mathbb{Z}^{<\mathbb{N}};$$
$$\mu(m\mathfrak{s}_H(g);n\mathfrak{s}_H(h))=\lambda(n;\mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g);\mathfrak{s}_H(h)),\quad m,n\in M,\ g,h\in N.$$

Proof. In the crossed extension $E\in \mathrm{Xext}(H_{\mathrm{m}},L,M,\mathbb{T})$ associated with $(\lambda,\mu)\in \mathrm{Z}(H_{\mathrm{m}},L,M,\mathbb{T})$:

$$1 \longrightarrow \mathbb{T} \longrightarrow E \longrightarrow L \longrightarrow 1,$$

we redefine the cross-section u in the following way:

$$u(m\mathfrak{s}_H(g))=u(m)u(\mathfrak{s}_H(g)),\quad m\in M, g\in N,$$

so that $\mu(m;\mathfrak{s}_H(g))=1, m\in M, g\in N$. We now compute, for $m,n\in M, h\in \mathbb{Z}^{<\mathbb{N}}$:

$$\begin{aligned}\lambda(m;n\mathfrak{s}_H(h))u(m)&=\alpha_{n\mathfrak{s}_H(h)}(u(m))=u(n)\alpha_{\mathfrak{s}_H(h)}(u(m))u(n)^{-1}\\&=\lambda(m;\mathfrak{s}_H(h))u(mn)u(n)^{-1}\\&=\lambda(m;\mathfrak{s}_H(h))u(m),\end{aligned}$$

for $g,h\in N$, we continue the computation:

$$\begin{aligned}&\mu(m\mathfrak{s}_H(g);n\mathfrak{s}_H(h))u(m\mathfrak{s}_H(g)n\mathfrak{s}_H(h))\\&\qquad=u(m\mathfrak{s}_H(g))u(n\mathfrak{s}_H(h))\\&\qquad=u(m)u(\mathfrak{s}_H(g))u(n)u(\mathfrak{s}_H(h)))\\&\qquad=u(m)\alpha_{\mathfrak{s}_H(g)}(u(n))u(\mathfrak{s}_H(g))u(\mathfrak{s}_H(h))\\&\qquad=\lambda(n;\mathfrak{s}_H(g))u(m)u(n)\mu(\mathfrak{s}_H(g);\mathfrak{s}_H(h))u(\mathfrak{s}_H(g)\mathfrak{s}_H(h))\\&\qquad=\lambda(n;\mathfrak{s}_H(g))u(mn)\mu(\mathfrak{s}_H(g);\mathfrak{s}_H(h))u(\mathfrak{s}_H(g)\mathfrak{s}_H(h))\\&\qquad=\lambda(n;\mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g);\mathfrak{s}_H(h))u(m\mathfrak{s}_H(g)n\mathfrak{s}_H(h)),\end{aligned}$$

and complete the proof. ♡

Groups $\boldsymbol{G}$, $\boldsymbol{H_{\mathrm{m}}}$, $\boldsymbol{G_{\mathrm{m}}}$, and $\boldsymbol{Q_{\mathrm{m}}}$: First, we fix notations. To work on the quotient group $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p, p \in \mathbb{N}, p \geq 2$, we set

$$\begin{aligned} &[i]_p = i + p\mathbb{Z} \in \mathbb{Z}_p; \quad i = np + \{i\}_p,\ 0 \leq \{i\}_p < p, \\ \eta_p\left([i]_p, [j]_p\right) = \{i\}_p + \{j\}_p - \{i+j\}_p = &\begin{cases} 0 & \text{if } \{i\}_p + \{j\}_p < p; \\ p & \text{if } \{i\}_p + \{j\}_p \geq p. \end{cases} \end{aligned} \tag{3.8}$$

We shall call the $p\mathbb{Z}$-valued cocycle $\eta_p \in \mathrm{Z}^2(\mathbb{Z}_p, p\mathbb{Z})$ the *Gauss cocycle*, which can be written in the following way:

$$\eta_p\left([i]_p, [j]_p\right) = p\left(\left[\frac{i+j}{p}\right] - \left[\frac{i}{p}\right] - \left[\frac{j}{p}\right]\right), \tag{3.8$'$}$$

where $[x], x \in \mathbb{R}$, is the Gauss symbol, i.e., the largest integer less than or equal to x.

Given a homomorphism m of the group G to $\mathbb{R}/T'\mathbb{Z}$ such that $\mathrm{Ker}(\mathrm{m}) \supset N$, we consider the group extension:

$$\begin{gathered} G_{\mathrm{m}} = \{(g, s) \in G \times \mathbb{R} : \dot{s}_{T'} = s + T'\mathbb{Z} = \mathrm{m}(g) \in \mathbb{R}/T'\mathbb{Z}\}, \\ 0 \longrightarrow \mathbb{Z} \xrightarrow{n \to z_0^n} G_{\mathrm{m}} \xrightarrow{\pi_{\mathrm{m}}} G \longrightarrow 1, \end{gathered}$$

where

$$z_0 = (0, T') \in G_{\mathrm{m}}.$$

Identifying m with $\mathrm{m}\circ\pi_0 \in \mathrm{Hom}(H, \mathbb{R}/T'\mathbb{Z})$, we also form a group extension:

$$\begin{aligned} H_{\mathrm{m}} &= \{(h, s) \in H \times \mathbb{R} : \mathrm{m}(h) = \dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}\} \\ &= \{(m, g, s) \in M \times G \times \mathbb{R} : \mathrm{m}(g) = \dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}\}, \end{aligned}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n \to z_0^n} H_{\mathrm{m}} \longrightarrow H \longrightarrow 1,$$

where the central element

$$z_0 = (1, T') \in H_{\mathrm{m}}$$

appears in both G_{m} and H_{m}. We hope that this abuse use of the same notation for two distinct elements in the different groups will not cause a headhach later: just like the zero elements in the ring theory.

By the assumption $N \subset \mathrm{Ker}(\mathrm{m})$, the homomophism m factors through the quotient group $Q = G/N$, so that it is also viewed as a homomorphism of $Q \mapsto \mathbb{R}/T'\mathbb{Z}$ and therefore we can form the group extension Q_{m} as before,

which sits on the following commutative diagram of exact sequences:

$$
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathfrak{s}_{\mathrm{m}}(N) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & G_{\mathrm{m}} \arrow[r, shift left, "\pi_{\mathrm{m}}"] \arrow[d, shift right, "\pi_Q"'] & G \arrow[r] \arrow[l, shift left, "\mathfrak{s}_{\mathrm{m}}"] \arrow[d, shift right, "\pi_Q"'] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & Q_{\mathrm{m}} \arrow[r, shift left, "\dot{\pi}_{\mathrm{m}}"] \arrow[d] \arrow[u, shift right, "\mathfrak{s}"'] & Q \arrow[r] \arrow[l, shift left, "\dot{\mathfrak{s}}_{\mathrm{m}}"] \arrow[d] \arrow[u, shift right, "\mathfrak{s}"'] & 1 \\
 & 0 & 1 & 1 &
\end{tikzcd}
$$

From the assumption $\mathrm{Ker}(\mathrm{m}) \supset N$, it follows that $\mathrm{m}(p_i a_i) = 0$, so that there exists an integer $q_i \in \mathbb{Z}, 0 \le q_i < p_i$ such that

$$
\begin{aligned}
\mathrm{m}_i = \{\mathrm{m}(a_i)\}_{T'} = \frac{q_i T'}{p_i} \in \left(\frac{T'}{p_i}\mathbb{Z}\right),\\
\mathrm{m}(a_i) = \dot{\mathrm{m}}_i = \mathrm{m}_i + \mathbb{T}'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}.
\end{aligned}
\tag{3.9}
$$

We set

$$
\left.
\begin{aligned}
z_i = (a_i, \mathrm{m}_i) \in G_{\mathrm{m}}, i \neq 0, \quad z_0 = (0, T') \in G_{\mathrm{m}},\\
\mathfrak{s}_{\mathrm{m}}(g) = \sum_{i \in \mathbb{N}} e_i(g) z_i = \left(g, \sum_{i\in\mathbb{N}} e_i(g)\mathrm{m}_i\right) = (g, \mathrm{n}(g)),\\
\mathrm{n}(g) = \sum_{i\in\mathbb{N}} e_i(g)\mathrm{m}_i, \quad g \in G.
\end{aligned}
\right\}
\tag{3.10}
$$

Then G_{m} decomposes in the following way:

$$
\left.
\begin{aligned}
G_{\mathrm{m}} = \mathbb{Z}z_0 \oplus \mathfrak{s}_{\mathrm{m}}(G) = \sum_{i\in\mathbb{N}_0}{}^{\oplus} \mathbb{Z} z_i, \text{ where } \mathbb{N}_0 = \mathbb{N} \cup \{0\},\\
\tilde{g} = \tilde{e}_0(\tilde{g}) z_0 + \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g}) z_i \in G_{\mathrm{m}};\\
\tilde{g} = (g, s) = (0, \tilde{e}_0(g,s)T') + \left(\sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g}) a_i, \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g})\mathrm{m}_i\right)\\
= (0, \tilde{e}_0(g,s)T') + \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g}) z_i;\\
\tilde{e}_0(g,s) = \frac{s - \mathrm{n}(g)}{T'} \in \mathbb{Z}, \quad \tilde{e}_i(g,s) = e_i(g), \quad i \in \mathbb{N}.
\end{aligned}
\right\}
\tag{3.11}
$$

In particular, if $g \in N$, we have

$$g = (g, 0) = -\frac{\mathrm{n}(g)}{T'} z_0 + \sum_{i \in \mathbb{N}} e_i(g) z_i,$$

so that

$$\tilde{e}_0(g) = -\frac{\mathrm{n}(g)}{T'} \neq 0 \quad \text{unless} \quad \mathrm{n}(g) = \sum_{i \in \mathbb{N}} e_i(g) \mathrm{m}_i = 0.$$

We then have

$$\mathrm{m}(g) = [\mathrm{n}(g)]_{T'} \in \mathbb{R}/T'\mathbb{Z}.$$

Setting

$$b_j = p_j a_j, \quad j \in \mathbb{N},$$

we write every $g \in N$ uniquely in the form:

$$g = \sum_{j \in \mathbb{N}} \frac{e_j(g)}{p_j} b_j = \sum_{j \in \mathbb{N}} e_{j,N}(g) b_j, \tag{3.12}$$

where

$$e_{j,N}(g) = \frac{e_j(g)}{p_j},$$

and also in H_m we have

$$b_j = p_j z_j - p_j m_j z_0 = p_j z_j - q_j z_0. \tag{3.12$'$}$$

Remark. The element $(a_i, 0)$ is NOT a member of G_m.

Next we define a cross-section $\dot{\mathfrak{s}}_\mathrm{m} : Q \mapsto Q_\mathrm{m}$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} G_\mathrm{m} & \xleftarrow{\mathfrak{s}_\mathrm{m}} & G \\ \uparrow{\scriptstyle \mathfrak{s}} & & \uparrow{\scriptstyle \mathfrak{s}} \\ Q_\mathrm{m} & \xleftarrow{\dot{\mathfrak{s}}_\mathrm{m}} & Q \end{array}$$

First, we set

$$\left.\begin{aligned} \dot{g} &= g + N \in Q = G/N, \quad g \in G; \\ \mathfrak{s}(q) &= \sum_{i \in \mathbb{N}} \{e_i(q)\}_{p_i} a_i, \quad q \in Q; \\ \dot{a}_i &= \pi_{Q_\mathrm{m}}(a_i), \quad \dot{z}_i = (\dot{a}_i, \mathrm{m}_i) \\ \dot{\mathfrak{s}}_\mathrm{m}(q) &= \sum_{i \in \mathbb{N}} \{e_i(q)\}_{p_i} \dot{z}_i = \sum_{i \in \mathbb{N}} \{e_i(q)\}_{p_i} \Big(\dot{a}_i, \mathrm{m}_i\Big) \\ &= \left(q, \sum_{i \in \mathbb{N}} \{e_i(q)\}_{p_i} \mathrm{m}_i\right); \\ \mathfrak{s}(q, s) &= (\mathfrak{s}(q), s) \in G_\mathrm{m}, \quad (q, s) \in Q_\mathrm{m}. \end{aligned}\right\} \tag{3.13}$$

The cross-section $\mathfrak{s} : Q_{\rm m} \mapsto G_{\rm m}$ gives rise to an N-valued cocycle:

$$\mathfrak{n}_{N} = \partial_{Q}\mathfrak{s} \in \mathrm{Z}^2(Q_{\rm m}, N), \tag{3.14}$$

which is given by:

$$\begin{aligned}
\mathfrak{n}_{N}(\tilde{q}_1;\tilde{q}_2) &= \mathfrak{s}(q_1, s_1) + \mathfrak{s}(q_2, s_2) - \mathfrak{s}(q_1 + q_2, s_1 + s_2)\\
&= (\mathfrak{s}(q_1) + \mathfrak{s}(q_2) - \mathfrak{s}(q_1 + q_2), 0)\\
&= \left(\sum_{i\in\mathbb{N}}\eta_{p_i}\Big([e_i(q_1)]_{p_i};[e_i(q_2)]_{p_i}\Big)a_i, 0\right)\\
&= \sum_{i\in\mathbb{N}}\Big(\eta_{p_i}\Big([e_i(q_1)]_{p_i};[e_i(q_2)]_{p_i}\Big)a_i, 0\Big) \in N = N\times\{0\}.
\end{aligned}$$

for each pair $\tilde{q}_1 = (q_1, s_1), \tilde{q}_2 = (q_2, s_2) \in Q_{\rm m}$.

For each element

$$h = (m, g) \in H, \quad m \in M, g \in G,$$

we write $m = m_0(h)$ and $g = \pi_{G}(h)$. Then we have

$$L = \pi_{G}^{-1}(N)$$

and

$$m_0(gh) = m_0(g) + m_0(h) + \mathfrak{n}_{M}(\pi_{G}(g);\pi_{G}(h)), \quad g, h \in H.$$

For short, we write:

$$e_{i,j}(\tilde{g}) = e_{i,j}(m_0(g)) \text{ for } \tilde{g} = (m_0(g), g, s) \in H_{\rm m}, i, j \in \mathbb{N}.$$

With

$$\mathfrak{s}_{H}(g) = (0, g) \in H \quad \text{for each } g \in G,$$

we have

$$\mathfrak{n}_{M}(g;h) = \mathfrak{s}_{H}(g) + \mathfrak{s}_{H}(h) - \mathfrak{s}_{H}(g+h) = \partial_{G}\mathfrak{s}_{H}(g;h),\ g, h \in G.$$

With

$$\dot{\mathfrak{s}} = \mathfrak{s}_{H}\circ\mathfrak{s},$$

we obtain a cross-section $\dot{\mathfrak{s}}$ of $\pi_{Q}\circ\pi_{G} : H \mapsto Q = H/L$, which gives rise to an L-valued second cocycle $\mathfrak{n}_{L} \in \mathrm{Z}^2(Q, L)$:

$$\begin{aligned}
\mathfrak{n}_{L}(q_1;q_2) &= \dot{\mathfrak{s}}(q_1)\dot{\mathfrak{s}}(q_2)\dot{\mathfrak{s}}(q_1+q_2)^{-1}, \quad q_1, q_2 \in Q,\\
&= \mathfrak{s}_{H}\Big(\mathfrak{s}(q_1)\Big)\mathfrak{s}_{H}\Big(\mathfrak{s}(q_2)\Big)\mathfrak{s}_{H}\Big(\mathfrak{s}(q_1+q_2)\Big)^{-1}\\
&= \mathfrak{n}_{M}\Big(\mathfrak{s}(q_1);\mathfrak{s}(q_2)\Big)\mathfrak{s}_{H}\Big(\mathfrak{s}(q_1)+\mathfrak{s}(q_2)\Big)\mathfrak{s}_{H}\Big(\mathfrak{s}(q_1+q_2)\Big)^{-1}\\
&= \mathfrak{n}_{M}\Big(\mathfrak{s}(q_1);\mathfrak{s}(q_2)\Big)\mathfrak{s}_{H}\Big(\mathfrak{n}_{N}(q_1;q_2)+\mathfrak{s}(q_1+q_2)\Big)\mathfrak{s}_{H}\Big(\mathfrak{s}(q_1+q_2)\Big)^{-1}\\
&= \mathfrak{n}_{M}\Big(\mathfrak{s}(q_1);\mathfrak{s}(q_2)\Big)\mathfrak{n}_{M}\Big(\mathfrak{n}_{N}(q_1;q_2);\mathfrak{s}(q_1+q_2)\Big)^{-1}\mathfrak{s}_{H}\Big(\mathfrak{n}_{N}(q_1;q_2)\Big).
\end{aligned} \tag{3.15}$$

We further compute the j,k-components and k-components:

$$\left.\begin{aligned} e_{j,k}\Big(\mathfrak{n}_M(\mathfrak{s}(q_1);\mathfrak{s}(q_2))\Big) &= e_j(\mathfrak{s}(q_1))e_k(\mathfrak{s}(q_2))\\ &= \{e_j(q_1)\}_{p_j}\{e_k(q_2)\}_{p_k},\\ e_{j,k}\Big(\mathfrak{n}_M(\mathfrak{n}_N(q_1;q_2);\mathfrak{s}(q_1+q_2)\Big) &= e_j(\mathfrak{n}_N(q_1;q_2))e_k(\mathfrak{s}(q_1+q_2))\\ &= \eta_{p_j}\Big([e_j(q_1)]_{p_j};[e_j(q_2)]_{p_j}\Big)\{e_k(q_1+q_2)\}_{p_k},\\ e_k\Big(\mathfrak{s}_H\Big(\mathfrak{n}_N(q_1;q_2)\Big)\Big) &= \eta_{p_k}\Big([e_k(q_1)]_{p_k};[e_k(q_2)]_{p_k}\Big). \end{aligned}\right\} \tag{3.16}$$

Since

$$H_{\mathrm{m}} = M \times_{\pi^*_{\mathrm{m}}(\mathfrak{n}_M)} \left(\sum_{i\in\mathbb{N}}{}^{\oplus}\mathbb{Z}z_i \oplus \mathbb{Z}z_0\right),$$

for each $h=(m,g)\in H$, we set

$$\mathfrak{s}_{\mathrm{m}}(h) = (m,\mathfrak{s}_{\mathrm{m}}(g)) = \left(m, \sum_{i\in\mathbb{N}} e_i(g)z_i\right) = \left(m, g, \sum_{i\in\mathbb{N}} e_i(g)\mathrm{m}_i\right), \tag{3.17}$$

and we identify $\ell=(m,Pg)\in L$ with $(m,Pg,0)\in H_{\mathrm{m}}$, so that L is a subgroup of H_{m}, while H is not.

§4. The Characteristic Cohomology Group $\boldsymbol{\Lambda(H_{\mathrm{m}},L,M,\mathbb{T})}$.

Since H is a universal resolution group for $G=\mathbb{Z}^{<\mathbb{N}}$, every third cohomology class $[c]\in \mathrm{H}^3(G,\mathbb{T})$ is of the form $[c]=\delta_{\mathrm{HJR}}[\lambda,\mu]$ for some $[\lambda,\mu]\in\Lambda(H,M,\mathbb{T})$. So every outer action $\dot\alpha$ of G on a factor $\mathcal{M}$ of type III_λ comes from an action α of H, i.e., the outer action $\dot\alpha$ is given by

$$\dot\alpha_g = \alpha_{\mathfrak{s}_H(g)},\quad g\in G. \tag{4.1}$$

But the action α of H does not give rise to an action of H on the reduced (discrete) core $\widetilde{\mathcal{M}}_d$. Instead, the action α of H on $\mathcal{M}$ gives rise naturally to an action, denoted by the same notation α, of H_{m} on $\widetilde{\mathcal{M}}_d$ where

$$\mathrm{m}(h) = \bmod(\alpha_h) \in \mathbb{R}/T'\mathbb{Z},\ h\in H.$$

If $N=\dot\alpha^{-1}(\mathrm{Cnt}_{\mathrm{r}}(\mathcal{M})))\subset G$, then $L=\alpha^{-1}(\mathrm{Cnt}_{\mathrm{r}}(\mathcal{M}))$. We make a basic assumption on the subgroup N:

$$N = PG = P\mathbb{Z}^{<\mathbb{N}}.$$

In the case that G is finitely generated free abelian group, the fundamental structure theorem for finitely generated abelian groups guarantees that every subgroup of G is of this form.

We study first the characteristic cohomology group $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$ and modified HJR-map $\delta : \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) \mapsto \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$.

We introduce a series of notations first:

$$\left.\begin{aligned} &\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \mathbb{Z}_+;\\ \Delta_0 = \{(i,j,k) \in \mathbb{N}_0^3 : i<j<k\} &\cup \{(i,i,k) \in \mathbb{N}_0^3; i<k\}\\ &\cup \{(k,i,k) \in \mathbb{N}_0^3 : i<k\},\\ &\Delta = \Delta_0 \cap \mathbb{N}^3. \end{aligned}\right\} \tag{4.2}$$

For each $g \in H_{\mathrm{m}}$, let $m_0(g)$ be the M-component of g, i.e.,

$$m_0(g) = g\mathfrak{s}_H(\pi_G(g))^{-1} \in M, \quad g \in H_{\mathrm{m}}. \tag{4.3}$$

We regard e_i and $e_{j,k}$ as functions defined on H_{m} by fixing the coordinate system:

$$\tilde{g} = \left(\sum_{1\le j<k} e_{j,k}(g)(a_j \wedge a_k), \sum_{i \in \mathbb{N}_0} \tilde{e}_i(\tilde{g}) z_i \right) \in H_{\mathrm{m}}, \text{ with } g = \pi_{\mathrm{m}}(\tilde{g}) \in H. \tag{4.4}$$

We then introduce a cochain $B_{jk} \in \mathrm{C}^1(H_{\mathrm{m}}, \mathbb{R})$ defined by the following:

$$B_{jk}(h) = \begin{cases} -e_{j,k}(m_0(h)) & \text{for } j<k;\\ -\dfrac{(e_je_j)(h)}{2} & \text{for } j=k;\\ e_{k,j}(m_0(h)) - (e_je_k)(h) & \text{for } j>k, \end{cases} \quad h \in H_{\mathrm{m}}. \tag{4.5}$$

The cochain enjoys the property:

$$\partial_H B_{jk} = \pi_0^*(e_j \otimes e_k) \quad \text{for } j,k \in \mathbb{N}. \tag{4.6}$$

We continue to define the following cochains for each $a \in \mathbb{R}^{\mathbb{N}_0^3}$:

$$\begin{aligned} X_a(i,j,k) &= a(i,j,k)e_{j,k}\otimes e_i + a(j,i,k)e_{i,k}\otimes e_j + a(k,i,j)e_{i,j}\otimes e_k,\\ X_a(i,k) &= a(i,i,k)e_{i,k} \otimes e_i + a(k,i,k)e_{i,k} \otimes e_k;\\ Y_a(i,j,k) &= a(i,j,k)\Big(B_{ij}\otimes e_k + e_k\otimes B_{ji} - B_{ik}\otimes e_j - e_j\otimes B_{ki}\Big)\\ &\quad + a(j,i,k)\Big(B_{ji}\otimes e_k + e_k\otimes B_{ij} - B_{jk}\otimes e_i - e_i\otimes B_{kj}\Big)\\ &\quad + a(k,i,j)\Big(B_{ki}\otimes e_j + e_j\otimes B_{ik} - B_{kj}\otimes e_i - e_i\otimes B_{jk}\Big),\\ Y_a(i,k) &= a(i,i,k)(B_{ii} \otimes e_k + e_k \otimes B_{ii} - B_{ik} \otimes e_i - e_i \otimes B_{ki})\\ &\quad + a(k,i,k)(B_{ki} \otimes e_k + e_k \otimes B_{ik} - B_{kk} \otimes e_i - e_i \otimes B_{kk}),\\ Z(\cdots)(g;h) &= Y(\cdots)(m_0(h);g);\\ Z_a(i,j,k) &= a(i,j,k)\Big(e_j\otimes e_{i,k} - e_k\otimes e_{i,j}\Big)\\ &\quad + a(j,i,k)\Big(e_k\otimes e_{i,j} + e_i\otimes e_{j,k}\Big) + a(k,i,j)\Big(e_j\otimes e_{i,k} - e_i\otimes e_{j,k}\Big), \end{aligned}$$

$$
\begin{aligned}
Z_a(i,k) &= a(i,i,k)e_i\otimes e_{i,k} + a(k,i,k)e_k\otimes e_{i,k};\\
f_{i,j,k} &= 2(e_ie_j)\otimes e_k - 3e_i\otimes(e_je_k) + e_j\otimes(e_ie_k)\\
&\qquad\qquad\qquad\qquad - 2(e_ie_k)\otimes e_j - e_k\otimes(e_ie_j),\\
U_a(i,j,k) &= \frac{1}{6}\Big(a(i,j,k)f_{i,j,k} + a(j,i,k)f_{j,i,k} + a(k,i,j)f_{k,i,j}\\
&\qquad\qquad\qquad - (\mathrm{AS}a)(i,j,k)f_{i,j,k}\Big),\\
U_a(i,k) &= -a(i,i,k)B_{ii}\otimes e_k + a(k,i,k)(B_{kk}\otimes e_i - e_k\otimes(e_ie_k)),\\
V_a(i,j,k) &= Z_a(i,j,k) + \pi_G^* U_a(i,j,k),\\
V_a(i,k) &= Z_a(i,k) + \pi_G^* U_a(i,k).
\end{aligned}
$$

The infinite summations:

$$
\left.
\begin{aligned}
X_a &= \sum_{i<j<k} X_a(i,j,k) + \sum_{i<k} X_a(i,k);\\
Y_a &= \sum_{i<j<k} Y_a(i,j,k) + \sum_{i<k} Y_a(i,k);\\
U_a &= \sum_{i<j<k} U_a(i,j,k) + \sum_{i<k} U_a(i,k);\\
V_a &= \sum_{i<j<k} V_a(i,j,k) + \sum_{i<k} V_a(i,k);\\
Z_a &= \sum_{i<j<k} Z_a(i,j,k) + \sum_{i<k} Z_a(i,k)
\end{aligned}
\right\}
\tag{4.7}
$$

will become all finite sums as soon as variables from M or H_{m} are fed in. So no divergence problem in the infinite sums will occur.

The cochain $f_{i,j,k}$ relates basic cocycles $e_i\otimes e_j\otimes e_k$ and the asymmetric tri-character:

$$
\begin{aligned}
\det{}_{ijk} &= (e_i\otimes e_j\otimes e_k + e_j\otimes e_k\otimes e_i + e_k\otimes e_i\otimes e_j)\\
&\qquad - (e_j\otimes e_i\otimes e_k + e_i\otimes e_k\otimes e_j + e_k\otimes e_j\otimes e_i)\\
&= e_i\wedge e_j\wedge e_k
\end{aligned}
$$

in the following way:

$$
\det{}_{ijk} = \partial_L f_{i,j,k} + 6e_i\otimes e_j\otimes e_k,\quad i<j<k, \tag{4.8}
$$

which can be confirmed by a direct computation.

Let Z be the set of all pairs (a,b) of functions a: $(i,j,k)\in\mathbb{N}^3\mapsto a(i,j,k)\in\mathbb{R}$ and b: $(i,j)\in\mathbb{N}_0^2\mapsto b(i,j)\in\mathbb{R}$ satisfying the following requirements:

a) The requirements on the parameter a is given by:

$$
\left.
\begin{aligned}
&a(i,j,k)=0 \text{ for } j\geq k \text{ and } a(0,j,k)=0 \quad \text{for every } j,k\in\mathbb{N}_0,\\
&\qquad(\mathrm{AS}a)(i,j,k) = a(i,j,k) - a(j,i,k) + a(k,i,j)\\
&\qquad\qquad\qquad \in \left(\frac{1}{\gcd(p_i,p_j,p_k)}\mathbb{Z}\right).
\end{aligned}
\right\}
\tag{4.9Z-a}
$$

b) The requirements for the parameter b is given by:

$$\left.\begin{array}{c} b(i,j)p_j - b(i,0)q_j \in \mathbb{Z} \quad \text{for} \quad i,j \in \mathbb{N}, i<j, \\ b(0,j)=0, \quad j \in \mathbb{N}_0. \end{array}\right\} \tag{4.9Z-b}$$

Let Z_a be the set of $a \in \mathbb{R}^{\mathbb{N}^3}$ satisfying the above requirement (4.9Z-a) and Z_b be the set of all $b \in \mathbb{R}^{\mathbb{N}_0^2}$ with the properties of (4.9Z-b). So we have

$$Z = Z_a \oplus Z_b.$$

Let B be the subgroup of Z consisting of all those $(a,b) \in Z$ such that

a) The coboundary condition on the parameter a is given by:

$$\left.\begin{array}{c} a(i,j,k), a(k,i,j), a(j,i,k) \in \mathbb{Z} \quad \text{if } i<j<k, \\ a(i,i,k) \in 2\mathbb{Z} \quad \text{if } i<k; \quad a(k,i,k) \in 2\mathbb{Z} \quad \text{if } i<k, \end{array}\right\} \tag{4.9B-a}$$

b) The coboundary condition on the parameter b is given by:

$$\left.\begin{array}{c} \dfrac{b(i,j)}{p_i} + \dfrac{b(j,i)}{p_j} \in \left(\dfrac{1}{p_i}\mathbb{Z}\right) + \left(\dfrac{1}{p_j}\mathbb{Z}\right) = \left(\dfrac{1}{\operatorname{lcm}(p_i,p_j)}\mathbb{Z}\right), i<j, \\ b(i,0) \in \mathbb{Z}, \quad b(i,i) \in \mathbb{Z}, \quad i \in \mathbb{N}. \end{array}\right\} \tag{4.9B-b}$$

Respectively, let B_a (resp. B_b) be the set of all those $b \in \mathbb{R}^{\mathbb{N}_0^2}$ satisfying the requirement of (4.9B-a) (resp. (4.9B-b)). Thus we have

$$B = B_a \oplus B_b.$$

and set

$$\Lambda = \Lambda_a \oplus \Lambda_b, \quad \Lambda_a = \mathrm{Z}_a/\mathrm{B}_a, \quad \Lambda_b = \mathrm{Z}_b/\mathrm{B}_b,$$
$$H_a = Z_a/B_a, \quad H_b = Z_b/B_b.$$

With $D(i,j,k) = \gcd(p_i, p_j, p_k)$ for each triplet $i<j<k, i,j,k \in \mathbb{N}$, we set

$$\mathrm{Z}_a(i,j,k) = \left\{(u,v,w) \in \mathbb{R}^3 : u - v + w \in \left(\frac{1}{D(i,j,k)}\mathbb{Z}\right)\right\},$$
$$\mathrm{B}_a(i,j,k) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$
$$\text{where } u = a(i,j,k),\ v = a(j,i,k),\ w = a(k,i,j).$$

For a pair $i<k, i,k \in \mathbb{N}$, we set

$$\mathrm{Z}_a(i,k) = \left\{(x,y) \in \mathbb{R}^2\right\} = \mathbb{R} \oplus \mathbb{R}, \quad \mathrm{B}_a(i,k) = (2\mathbb{Z}) \oplus (2\mathbb{Z}),$$

where $x = a(i, i, k)$ and $y = a(k, i, k)$. We then naturally define:

$$\begin{aligned}\Lambda_a(i,j,k) &= \mathrm{Z}_a(i,j,k)/\mathrm{B}_a(i,j,k)\\ &\cong \left(\left(\frac{1}{D(i,j,k)}\mathbb{Z}\right)\Big/\mathbb{Z}\right) \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}, \quad i<j<k;\\ \Lambda_a(i,k) &= \mathrm{Z}_a(i,k)/\mathrm{B}_a(i,k) = \mathbb{R}/(2\mathbb{Z}) \oplus \mathbb{R}/(2\mathbb{Z}), \quad i<k.\end{aligned}$$

Here the above second isomorphism can be seen easily by considering the matrix:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{Z}).$$

For each ordered pair $i < j, i, j \in \mathbb{N}$, we define

$$\left.\begin{array}{c} \mathrm{Z}_b(i,j) = \big\{(x,u,y,v) \in \mathbb{R}^4 : p_j x - q_j u \in \mathbb{Z}, p_i y - q_i v \in \mathbb{Z}\big\};\\ \mathrm{B}_b(i,j) = \{(x,u,y,v) \in \mathrm{Z}_b(i,j) : p_j x + p_i y \in D_{i,j}\mathbb{Z}, u, v \in \mathbb{Z}\},\\ \mathrm{Z}_b(i,i) = \big\{z = (x,u) \in \mathbb{R}^2 : p_i x - q_i u \in \mathbb{Z}\big\}, \quad \mathrm{B}_b(i,i) = \mathbb{Z} \oplus \mathbb{Z},\\ \Lambda_b(i,j) = \mathrm{Z}_b(i,j)/\mathrm{B}_b(i,j), \quad \Lambda_b(i,i) = \mathrm{Z}_b(i,i)/\mathrm{B}_b(i,i), \end{array}\right\} \tag{4.10}$$

where

$$D_{i,j} = \gcd(p_i, p_j).$$

Definition 4.1. To each $(a, b) \in Z$ we associate a cochain $(\lambda_{a,b}, \mu_a)$ defined by the following:

$$\left.\begin{aligned}\lambda_{a,b}(g;h) &= \exp(2\pi\mathrm{i}((Y_a + X_{\mathrm{AS}a})(g;h)))\\ &\qquad\times \exp\left(2\pi\mathrm{i}\left(\sum_{i\in\mathbb{N}, j\in\mathbb{N}_0} b(i,j)e_{i,N}(g)\tilde{e}_j(h)\right)\right),\\ \eta_a(g;h) &= \exp(2\pi\mathrm{i}(Y_a(g;h))),\\ \mu_a(g;h) &= \exp\Big(2\pi\mathrm{i}V_a(g;h)\Big)\\ &= \lambda_{a,b}(m_0(h);g)\exp\Big(2\pi\mathrm{i}U_a(\pi_G(g);\pi_G(h))\Big),\end{aligned}\right\} \tag{4.11}$$

for each $(g, h) \in L \times H_{\mathrm{m}}$. In the case that $b = 0$ (resp. $a = 0$) we denote the corresponding cochains by (λ_a, μ_a) (resp. λ_b).

Theorem 4.2. a) *The cochain (λ_a, μ_a) is a characteristic cocycle in $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ and the correspondence $a \in Z_a \mapsto (\lambda_a, \mu_a) \in \mathrm{Z}_a$ gives the following commutative diagram of exact sequences*:

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & B_a \arrow[r] \arrow[d] & a \in Z_a \arrow[r] \arrow[d] & {[a] \in H_a} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{B}_a \arrow[r] & (\lambda_a, \mu_a) \in \mathrm{Z}_a \arrow[r] & {[\lambda_a, \mu_a] \in \Lambda_a} \arrow[r] \arrow[d] & 1 \\ & & & 0 & \end{tikzcd} \tag{4.12-a}$$

b) *The correspondence* $b \in Z_b \mapsto \lambda_b \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ *gives the following commutative diagram of exact sequences*:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & B_b & \longrightarrow & b \in Z_b & \longrightarrow & [b] \in H_b & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{B}_b & \longrightarrow & (\lambda_b, 1) \in \mathrm{Z}_b & \longrightarrow & [\lambda_b] \in \Lambda_b & \longrightarrow & 1 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{4.12-b}$$

c) *The characteristic cohomology group*

$$\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) = \Lambda_a \oplus \Lambda_b$$

has further fine structure:

i) *The group* Λ_a *has the Cartesian product decomposition*:

$$\left.\begin{aligned} \Lambda_a = \prod_{i<j<k} \Lambda_a(i,j,k) \oplus \prod_{i<j} \Lambda_a(i,j); \\ \Lambda_a(i,j,k) \cong \mathbb{Z}_{D(i,j,k)} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}, \\ D(i,j,k) = \gcd(p_i, p_j, p_k); \\ \Lambda_a(i,j) \cong \mathbb{R}/(2\mathbb{Z}) \oplus \mathbb{R}/(2\mathbb{Z}). \end{aligned}\right\} \tag{4.13-a}$$

ii) *The group* Λ_b *has the fiber product decomposition into the family* $\{\Lambda_b(i,j) : i, j \in \mathbb{N}\}$ *and each group* $\Lambda_b(i,j)$ *is described as follows*:

$$\left.\begin{aligned} \Lambda_b(i,j) \cong \mathbb{Z}/(\gcd(p_i, p_j, q_i, q_j)\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}), i < j, \\ \Lambda_b(i,i) \cong \mathbb{Z}/(\gcd(p_i, q_i)\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}). \end{aligned}\right\} \tag{4.13-b}$$

The group $\Lambda_b(i,j)$ *and* (*resp.* $\Lambda_b(i,i)$) *is equipped with three* (*resp. one*) *homomorphisms*:

$$\left.\begin{aligned} \pi_{ij} : \Lambda_b(i,j) \mapsto \left(\frac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z}, \\ \pi^i_{i,j} : \Lambda_b(i,j) \mapsto \mathbb{R}/\mathbb{Z}, \quad \pi^j_{i,j} : \Lambda_b(i,j) \mapsto \mathbb{R}/\mathbb{Z}, \\ \pi^i_i : \Lambda_b(i,i) \mapsto \mathbb{R}/\mathbb{Z}, \end{aligned}\right\} \tag{4.14}$$

such that for each $z = (x, u, y, v) \in \mathrm{Z}_b(i,j)$

$$\left.\begin{aligned} \pi_{ij}([\lambda_z]) = [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}}, \\ \pi^i_{i,j}([\lambda_z]) = [u]_{\mathbb{Z}}, \quad \pi^j_{i,j}([\lambda_z]) = [v]_{\mathbb{Z}}, \\ \pi_{ii}([\lambda_z]) = [p_i x - q_i u]_{\mathbb{Z}}, \quad \pi^i_i([\lambda_z]) = [u]_{\mathbb{Z}}, \end{aligned}\right\} \tag{4.15}$$

where

$$\left.\begin{aligned} D(i,j) &= \gcd(p_i, p_j, q_i, q_j),\\ D_{i,j} = \gcd(p_i, p_j), &\quad E_{i,j} = \gcd(q_i, q_j),\\ r_{i,j} = \frac{p_i}{D_{i,j}}, \quad r_{j,i} = \frac{p_j}{D_{i,j}} &\quad s_{i,j} = \frac{q_i}{E_{i,j}}, \quad s_{j,i} = \frac{q_j}{E_{i,j}},\\ m_{i,j} = \frac{D_{i,j}}{D(i,j)}, &\quad n_{i,j} = \frac{E_{i,j}}{D(i,j)},\\ q_i w_{i,j} + q_j w_{j,i} = E_{i,j}, &\quad x_{i,j} D_{i,j} + y_{i,j} E_{i,j} = D(i,j). \end{aligned}\right\} \tag{4.16}$$

The group Λ_b is the fiber product of $\{\Lambda_b(i,j) : i,j \in \mathbb{N}\}$ relative to the maps $\left\{\pi^i_{i,j}, \pi^j_{i,j}, \pi^i_i : i,j \in \mathbb{N}\right\}$ in the sense that Λ_b is the group of all those $\lambda_b \in \prod_{(i,j)\in\mathbb{N}^2} \Lambda_b(i,j)$ such that

$$\pi^i_{i,j}[\lambda_b(i,j)] = \pi^i_i[\lambda_b(i,i)] = \pi^i_{ki}[\lambda_b(k,i)], \quad i,j,k \in \mathbb{N}. \tag{4.17}$$

We will prove the theorem in several steps.
First, we observe that the asymmetrization of $f_{i,j,k}$ is given by:

$$\begin{aligned} \mathrm{AS} f_{i,j,k} &= 2(e_i e_j) \wedge e_k - 3e_i \wedge (e_j e_k) + e_j \wedge (e_i e_k)\\ &\qquad\qquad - 2(e_i e_k) \wedge e_j - e_k \wedge (e_i e_j)\\ &= 3\Big((e_j e_k) \wedge e_i - (e_i e_k) \wedge e_j + (e_i e_j) \wedge e_k\Big). \end{aligned} \tag{4.18}$$

Lemma 4.3. i) *The difference of X_a and Y_a on $M \times H_{\mathrm{m}}$ is given by*:

$$X_a - Y_a = X_{\mathrm{AS}a} \quad \text{on} \quad M \times H_{\mathrm{m}}.$$

In particular if the following integers

$$e_{i,j}(m) e_k(g),\ e_{j,k}(m) e_i(g),\ e_{i,k}(m) e_j(g),\ e_{j,k}(m) e_i(g)$$

are all divisible by $\gcd(p_i, p_j, p_k)$, then we have for each $a \in Z$

$$Y_a(i,j,k)(m;g) \equiv X_a(i,j,k)(m;g) \mod \mathbb{Z}, \quad m \in M, g \in H_{\mathrm{m}}.$$

Therefore, if either $g \in L$ or $m \in L \wedge H_{\mathrm{m}}$, then the following congruence holds:

$$\begin{aligned} X_a(i,j,k)(m;g) &\equiv Y_a(i,j,k)(m;g) \mod \mathbb{Z};\\ X_a(i,j,k)(h_1 \wedge g; h_2) &\equiv Y_a(i,j,k)(h_1 \wedge g; h_2) \mod \mathbb{Z} \end{aligned} \tag{4.19}$$

for each $h_1, h_2 \in H_{\mathrm{m}}$.

ii) *For every* $m \in M$ *and* $g \in H_{\mathrm{m}}$ *and* $i < k$ *we have*

$$X_a(i,k)(m;g) = Y_a(i,k)(m;g). \tag{4.20}$$

Proof. i) We simply compute for $i < j < k$:

$$\begin{aligned}
&(X_a(i,j,k) - Y_a(i,j,k))(m;g)\\
&\quad = a(i,j,k)e_{j,k}(m)e_i(g) + a(j,i,k)e_{i,k}(m)e_j(g)\\
&\qquad\qquad + a(k,i,j)e_{i,j}(m)e_k(g)\\
&\qquad - a(i,j,k)\Big(e_{i,k}(m)e_j(g) - e_{i,j}(m)e_k(g)\Big)\\
&\qquad\quad - a(j,i,k)\Big(e_{i,j}(m)e_k(g) + e_{j,k}(m)e_i(g)\Big)\\
&\qquad\qquad - a(k,i,j)\Big(e_{i,k}(m)e_j(g) - e_{j,k}(m)e_i(g)\Big)\\
&\quad = \Big(a(i,j,k) - a(j,i,k) + a(k,i,j)\Big)\\
&\qquad\quad \times \Big(e_{j,k}(m)e_i(g) - e_{i,k}(m)e_j(g) + e_{i,j}(m)e_k(g)\Big).
\end{aligned}$$

Thus we conclude

$$(X_a - Y_a)(m;g) = X_{\mathrm{AS}a}(m;g), \quad m \in M, g \in H_{\mathrm{m}}.$$

ii) The assertion follows from an easy direct computation. ♡

Lemma 4.4. *If* $a \in \mathbb{R}^{\Delta}$ *is asymmetric modulo* $\left(\dfrac{1}{p_ip_jp_k}\mathbb{Z}\right)$ *in the sense that*:

$$(\mathrm{AS}a)(i,j,k) = a(i,j,k) - a(j,i,k) + a(k,i,j) \in \left(\frac{1}{p_ip_jp_k}\mathbb{Z}\right) \tag{4.21}$$

for each triplet $i < j < k$*, then the cochain* μ_a *of* (4.11)*, i.e.,*

$$\mu_a(g;h) = \exp(2\pi\mathrm{i}(V_a(g;h))), \quad g,h \in L,$$

is a second cocycle $\mu_a \in \mathrm{Z}^2(L,\mathbb{T})$.

Proof. Observing

$$(\partial_L\mu_a)(g_1;g_2;g_3) = \exp(2\pi\mathrm{i}(\partial_L V_a(g_1;g_2;g_3))), \quad g_1,g_2,g_3 \in L,$$

we compute the coboundary of V_a:

$$\begin{aligned}
&\partial_L V_a(i,j,k) = \partial_L Z_a(i,j,k) + \partial_L U_a(i,j,k)\\
&\quad = a(i,j,k)(e_j\otimes e_i\otimes e_k - e_k\otimes e_i\otimes e_j)\\
&\qquad\quad + a(j,i,k)(e_k\otimes e_i\otimes e_j + e_i\otimes e_j\otimes e_k)\\
&\qquad\qquad + a(k,i,j)(e_j\otimes e_i\otimes e_k - e_i\otimes e_j\otimes e_k)\\
&\qquad\quad + \frac{1}{6}\partial_L\Big(a(i,j,k)f_{i,j,k} + a(j,i,k)f_{j,i,k} + a(k,i,j)f_{k,i,j}\\
&\qquad\qquad\qquad - (\mathrm{AS}a)(i,j,k)f_{i,j,k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= a(i,j,k)(e_j\otimes e_i\otimes e_k - e_k\otimes e_i\otimes e_j)\\
&\qquad + a(j,i,k)(e_k\otimes e_i\otimes e_j + e_i\otimes e_j\otimes e_k)\\
&\qquad\qquad + a(k,i,j)(e_j\otimes e_i\otimes e_k - e_i\otimes e_j\otimes e_k)\\
&\qquad + \frac{1}{6}\Big(a(i,j,k)(\det_{ijk} - 6e_i\otimes e_j\otimes e_k)\\
&\qquad\quad + a(j,i,k)(\det_{jik} - 6e_j\otimes e_i\otimes e_k)\\
&\qquad\qquad + a(k,i,j)(\det_{kij} - 6e_k\otimes e_i\otimes e_j)\\
&\qquad\qquad\quad - (\mathrm{AS}a)(i,j,k)(\det_{ijk} - 6e_i\otimes e_j\otimes e_k)\Big)\\
&\equiv -(\mathrm{AS}a)(i,j,k)\Big(e_i\otimes e_j\otimes e_k - e_j\otimes e_i\otimes e_k + e_k\otimes e_i\otimes e_j\Big)\\
&\equiv 0 \quad \mod \quad \mathbb{Z} \quad \text{on } L\times L\times L,
\end{aligned}
$$

since $e_i\otimes e_j\otimes e_k$ takes values in $p_ip_jp_k\mathbb{Z}$ on $L\times L\times L$. Also we have

$$
\begin{aligned}
\partial_L V_a(i,k) &= \partial_L Z_a(i,k) + \partial_L U_a(i,k)\\
&= a(i,i,k)e_i\otimes e_i\otimes e_k + a(k,i,k)e_k\otimes e_i\otimes e_k - a(i,i,k)e_i\otimes e_i\otimes e_k\\
&\qquad + a(k,i,k)\Big(e_k\otimes e_k\otimes e_i - e_k\otimes(e_i\otimes e_k + e_k\otimes e_i)\Big)\\
&= 0.
\end{aligned}
$$

Hence μ_a is a second cocycle on L. ♡

Lemma 4.5. i) *For every $(a,b)\in \mathrm{Z}$, the pair $\{\lambda_{a,b},\mu_a\}$ is a characteristic cocycles in $\mathrm{Z}(H_{\mathrm{m}},L,M,\mathbb{T})$.*

ii) *Every characteristic cocycle $(\lambda,\mu)\in \mathrm{Z}(H_{\mathrm{m}},L,M,\mathbb{T})$ is cohomologous to some $(\lambda_{a,b},\mu_a)$.*

iii) *The characteristic cocycle $\{\lambda_{a,b},\mu_a\}\in \mathrm{Z}(H_{\mathrm{m}},L,M,\mathbb{T})$ is a coboundary if and only if $(a,b)\in \mathrm{B}$.*

Proof. i) We first check the cocycle identities for each $g,g_1,g_2\in L$ and $h,h_1,h_2\in H_{\mathrm{m}}$:

$$
\begin{aligned}
\Big(\Big(\partial_L\otimes\mathrm{id}\Big)\lambda_{a,b}\Big)(g_1;g_2;h) &= \frac{\mu_a(h^{-1}g_1h;h^{-1}g_2h)}{\mu_a(g_1;g_2)}\\
&= \lambda_{a,b}(g_2\wedge h;g_1); \qquad \text{(a)}
\end{aligned}
$$

$$
\begin{aligned}
\Big(\Big(\mathrm{id}\otimes\partial_{H_{\mathrm{m}}}\Big)\lambda_{a,b}\Big)(g;h_1;h_2) &= \frac{1}{\lambda_{a,b}(g\wedge h_1;h_2)}\\
&= \lambda_{a,b}(h_1\wedge g;h_2); \qquad \text{(b)}
\end{aligned}
$$

$$
\lambda_{a,b}(g;h) = \frac{\mu_a(h;h^{-1}gh)}{\mu_a(g;h)}, \quad g,h\in L. \qquad \text{(c)}
$$

Second, we compute for $g_1,g_2\in L$ and $h\in H_{\mathrm{m}}$:

$$
\begin{aligned}
&X_a(i,j,k)(g_2\wedge h;g_1)\\
&\qquad = a(i,j,k)e_{j,k}(g_2\wedge h)e_i(g_1) + a(j,i,k)e_{i,k}(g_2\wedge h)e_j(g_1)\\
&\qquad\qquad + a(k,i,j)e_{i,j}(g_2\wedge h)e_k(g_1)
\end{aligned}
$$

$$\begin{aligned}
&= a(i,j,k)e_i(g_1)\Big(e_j(g_2)e_k(h) - e_k(g_2)e_j(h)\Big)\\
&\qquad + a(j,i,k)e_j(g_1)\Big(e_i(g_2)e_i(h) - e_k(g_2)e_i(h)\Big)\\
&\qquad + a(k,i,j)e_k(g_1)\Big(e_i(g_2)e_j(h) - e_j(g_2)e_i(h)\Big)\\
&= \Big[a(i,j,k)e_i{\otimes}\Big(e_j{\otimes}e_k - e_k{\otimes}e_j\Big) + a(j,i,k)e_j{\otimes}\Big(e_i{\otimes}e_i - e_k{\otimes}e_i\Big)\\
&\qquad + a(k,i,j)e_k{\otimes}\Big(e_i{\otimes}e_j - e_j{\otimes}e_i\Big)\Big]\Big(g_1; g_2; h\Big).
\end{aligned}$$

On the other hand, we have

$$\left.\begin{aligned}
\Big(\Big(\partial_L{\otimes}\mathrm{id}\Big)Y_a(i,j,k)\Big) &= a(i,j,k)\Big(e_i{\otimes}e_j{\otimes}e_k - e_i{\otimes}e_k{\otimes}e_j\Big)\\
&\qquad + a(j,i,k)\Big(e_j{\otimes}e_i{\otimes}e_k - e_j{\otimes}e_k{\otimes}e_i\Big)\\
&\qquad\quad + a(k,i,j)\Big(e_k{\otimes}e_i{\otimes}e_j - e_k{\otimes}e_j{\otimes}e_i\Big).
\end{aligned}\right\} \tag{4.22}$$

Since

$$X_{\mathrm{ASa}}(i,j,k)(g_2 \wedge h; g_1) \equiv 0 \mod \mathbb{Z},$$

Lemma 4.3 yields, for each $g_1, g_2 \in L, h \in H_{\mathrm{m}}$, the following:

$$\begin{aligned}
\Big((\partial_L{\otimes}\mathrm{id})Y_a(i,j,k)\Big)(g_1; g_2; h) &= X_a(i,j,k)(g_2 \wedge h; g_1)\\
&\equiv Y_a(i,j,k)(g_2 \wedge h; g_1) \mod \mathbb{Z}.
\end{aligned}$$

Similarly, we have

$$((\partial_L{\otimes}\mathrm{id})Y_a(i,k))(g_1, g_2; h) \equiv Y_a(i,k)(g_2 \wedge h; g_1) \mod \mathbb{Z}.$$

Next, we have

$$\left.\begin{aligned}
X_a(i,j,k)(h_1 \wedge g; h_2) &= a(i,j,k)e_{j,k}(h_1 \wedge g)e_i(h_2)\\
&\qquad + a(j,i,k)e_{i,k}(h_1 \wedge g)e_j(h_2) + a(k,i,j)e_{i,j}(h_1 \wedge g)e_k(h_2)\\
&= \Big[a(i,j,k)\Big(e_k{\otimes}e_j - e_j{\otimes}e_k\Big){\otimes}e_i + a(j,i,k)\Big(e_k{\otimes}e_i - e_i{\otimes}e_k\Big){\otimes}e_j\\
&\qquad + a(k,i,j)\Big(e_j{\otimes}e_i - e_i{\otimes}e_j\Big){\otimes}e_k\Big]\Big(g; h_1; h_2\Big);\\
\Big(\mathrm{id}{\otimes}\partial_{H_{\mathrm{m}}}\Big)Y_a(i,j,k)(g; h_1; h_2) &= \Big[a(i,j,k)\Big(e_k{\otimes}e_j{\otimes}e_i - e_j{\otimes}e_k{\otimes}e_i\Big)\\
&\qquad + a(j,i,k)\Big(e_k{\otimes}e_i{\otimes}e_j - e_i{\otimes}e_k{\otimes}e_j\Big)\\
&\qquad + a(k,i,j)\Big(e_j{\otimes}e_i{\otimes}e_k - e_i{\otimes}e_j{\otimes}e_k\Big)\Big]\Big(g; h_1; h_2\Big)\\
&= X_a(i,j,k)(h_1 \wedge g; h_2)
\end{aligned}\right\} \tag{4.23}$$

and

$$\Big(\mathrm{id}{\otimes}\partial_{H_{\mathrm{m}}}\Big)X_{\mathrm{ASa}}(i,j,k) = 0.$$

Hence Lemma 4.3 again yields, for each $g \in L, h_1, h_2 \in H_{\mathrm{m}}$, that:

$$\begin{aligned}\Big(\mathrm{id}\otimes\partial_{H_{\mathrm{m}}}\Big)\Big(Y_a(i,j,k) + X_{\mathrm{AS}a}(i,j,k)\Big)(g;h_1;h_2)\\ \equiv \Big(Y_a(i,j,k) + X_{\mathrm{AS}a}(i,j,k)\Big)(h_1\wedge g;h_2) \mod \mathbb{Z}.\end{aligned}$$

Similarly, we get the following:

$$\begin{gathered}((\mathrm{id}\otimes\partial_{H_{\mathrm{m}}})Y_a(i,k))(g,h_1;h_2) = Y_a(i,k)(h_1\wedge g;h_2), \quad g\in L, h_1,h_2\in H_{\mathrm{m}},\\ X_{\mathrm{AS}a}(i,k) = 0.\end{gathered}$$

Thus so far we have established the formulae (a) and (b).

Now we move on to (c). Fixing $g, h \in L$, we compute the right hand side of (c):

$$\begin{aligned}\frac{\mu_a(h;h^{-1}gh)}{\mu_a(g;h)} &= \frac{\mu_a(h;(g\wedge h)g)}{\mu_a(g;h)} = \lambda_a(g\wedge h;h)\frac{\mu_a(h;g)}{\mu_a(g;h)}\\ &= \lambda_a(g\wedge h;h)\frac{\mu_a\Big(m_0(h)\mathfrak{s}_H(h);m_0(g)\mathfrak{s}_H(g)\Big)}{\mu_a\Big(m_0(g)\mathfrak{s}_H(g);m_0(h)\mathfrak{s}_H(h)\Big)}\\ &= \exp(2\pi\mathrm{i}(X_a(g\wedge h;h)))(\exp(2\pi\mathrm{i}(\mathrm{AS}V_a(h;g))))\\ &= \exp(2\pi\mathrm{i}((Y_a + X_{\mathrm{AS}a})(g\wedge h;h)))(\exp(2\pi\mathrm{i}(\mathrm{AS}V_a(h;g))).\end{aligned}$$

Next we prove the following:

$$\lambda_a(\mathfrak{s}_H(g);\mathfrak{s}_H(h)) = \lambda_a(g\wedge h;h)(\mathrm{AS}\mu_a)(\mathfrak{s}_H(h);\mathfrak{s}_H(g)), \quad g,h\in N.$$

First we observe that

$$X_{\mathrm{AS}a}(g;h) \equiv 0 \quad \mod \mathbb{Z} \quad \text{for} \quad g,h\in L.$$

So for the proof of (c), the term $X_{\mathrm{AS}a}$ can be ignored. With this fact in mind, we compute:

$$\begin{aligned}&X_a(i,j,k)(g\wedge h;h)\\ &\quad = a(i,j,k)e_{j,k}(g\wedge h)e_i(h) + a(j,i,k)e_{i,k}(g\wedge h)e_j(h)\\ &\qquad + a(k,i,j)e_{i,j}(g\wedge h)e_k(h)\\ &\quad = \Big[a(i,j,k)\Big(e_j\otimes(e_ke_i) - e_k\otimes(e_je_i)\Big)\\ &\qquad + a(j,i,k)\Big(e_i\otimes(e_ke_j) - e_k\otimes(e_ie_j)\Big)\\ &\qquad + a(k,i,j)\Big(e_i\otimes(e_je_k) - e_j\otimes(e_ie_k)\Big)\Big]\Big(g;h\Big),\end{aligned}$$

and also

$$\begin{aligned}
X_a(i,k)(g\wedge h;h) &= a(i,i,k)\Big(e_i(g)e_k(h)-e_k(g)e_i(h)\Big)e_i(h)\\
&\qquad + a(k,i,k)\Big(e_i(g)e_k(h)-e_k(g)e_i(h)\Big)e_k(h)\\
&= a(i,i,k)\Big(e_i\otimes(e_ie_k)-e_k\otimes e_i^2\Big)\Big(g;h\Big)\\
&\qquad + a(k,i,k)\Big(e_i\otimes e_k^2-e_k\otimes(e_ie_k)\Big)\Big(g;h\Big).
\end{aligned}$$

Next we determine the asymmetrization of $U_a(i,j,k)$ based on (4.18):

$$\begin{aligned}
&\mathrm{AS}U_a(i,j,k)\\
&\quad=\frac{1}{6}\Big(a(i,j,k)\mathrm{AS}f_{i,j,k}+a(j,i,k)\mathrm{AS}f_{j,i,k}+a(k,i,j)\mathrm{AS}f_{k,i,j}\\
&\qquad\qquad -(\mathrm{AS}a)(i,j,k)\mathrm{AS}f_{i,j,k}\Big)\\
&\quad=\frac{1}{2}\Big[a(i,j,k)\Big((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k\Big)\\
&\qquad + a(j,i,k)\Big((e_ie_k)\wedge e_j-(e_je_k)\wedge e_i+(e_ie_j)\wedge e_k\Big)\\
&\qquad + a(k,i,j)\Big((e_ie_j)\wedge e_k-(e_je_k)\wedge e_i+(e_ie_k)\wedge e_j\Big)\\
&\qquad - \Big(a(i,j,k)-a(j,i,k)+a(k,i,j)\Big)\\
&\qquad\qquad \times\Big((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k\Big)\Big]\\
&\quad=\frac{1}{2}\Big[a(j,i,k)\Big((e_ie_k)\wedge e_j-(e_je_k)\wedge e_i+(e_ie_j)\wedge e_k\Big)\\
&\qquad + a(k,i,j)\Big((e_ie_j)\wedge e_k-(e_je_k)\wedge e_i+(e_ie_k)\wedge e_j\Big)\\
&\qquad + \Big(a(j,i,k)-a(k,i,j)\Big)\\
&\qquad\qquad \times\Big((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k\Big)\Big]\\
&\quad=-a(k,i,j)(e_je_k)\wedge e_i+a(k,i,j)(e_ie_k)\wedge e_j\\
&\qquad + a(j,i,k)(e_ie_j)\wedge e_k.
\end{aligned}$$

Hence we get

$$\left.\begin{aligned}
\mathrm{AS}U_a(i,j,k) &= -a(k,i,j)\Big((e_je_k)\otimes e_i-e_i\otimes(e_je_k)\Big)\\
&\qquad + a(k,i,j)\Big((e_ie_k)\otimes e_j-e_j\otimes(e_ie_k)\Big)\\
&\qquad\qquad + a(j,i,k)\Big((e_ie_j)\otimes e_k-e_k\otimes(e_ie_j)\Big).
\end{aligned}\right\}\tag{4.24}$$

We also check the asymmetrization of $U_a(i,k)$:

$$\begin{aligned}
\mathrm{AS}U_a(i,k) &= a(i,i,k)e_k\wedge B_{ii}+a(k,i,k)(B_{kk}\wedge e_i-e_k\wedge(e_ie_k))\\
&= \frac{a(i,i,k)}{2}\Big(e_i^2\otimes e_k-e_k\otimes e_i^2\Big)+\frac{a(k,i,k)}{2}\Big(e_i\otimes e_k^2-e_k^2\otimes e_i\Big)\\
&\qquad + a(k,i,k)\Big((e_ie_k)\otimes e_k-e_k\otimes(e_ie_k)\Big).
\end{aligned}$$

We then combine these with the above computations for $X_a(i,j,k)$, paying attention to the order of variables in the first term and the second term[1]:

$$
\begin{aligned}
&X_a(i,j,k)(g\wedge h;h)+\mathrm{ASU}_a(i,j,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\
&\quad= a(i,j,k)\Big(e_j\otimes(e_ke_i)-e_k\otimes(e_je_i)\Big)\\
&\qquad\qquad+a(j,i,k)\Big(e_i\otimes(e_ke_j)-e_k\otimes(e_ie_j)\Big)\\
&\qquad\qquad\qquad+a(k,i,j)\Big(e_i\otimes(e_je_k)-e_j\otimes(e_ie_k)\Big)\\
&\qquad\qquad+\Big(a(k,i,j)(e_je_k)\wedge e_i-a(k,i,j)(e_ie_k)\wedge e_j\\
&\qquad\qquad\qquad-a(j,i,k)(e_ie_j)\wedge e_k\Big)\\
&\qquad\quad= a(i,j,k)\Big(e_j\otimes(e_ke_i)-e_k\otimes(e_je_i)\Big)\\
&\qquad\qquad+a(j,i,k)\Big(e_i\otimes(e_ke_j)-e_k\otimes(e_ie_j)-(e_ie_j)\wedge e_k\Big)\\
&\qquad\qquad+a(k,i,j)\Big(e_i\otimes(e_je_k)-e_j\otimes(e_ie_k)+(e_je_k)\wedge e_i\\
&\qquad\qquad\qquad-(e_ie_k)\wedge e_j\Big)\\
&\qquad\quad= a(i,j,k)\Big(e_j\otimes(e_ke_i)-e_k\otimes(e_je_i)\Big)\\
&\qquad\qquad+a(j,i,k)\Big(e_i\otimes(e_ke_j)-e_k\otimes(e_ie_j)\\
&\qquad\qquad\qquad-(e_ie_j)\otimes e_k+e_k\otimes(e_ie_j)\Big)\\
&\qquad\qquad+a(k,i,j)\Big(e_i\otimes(e_je_k)-e_j\otimes(e_ie_k)\\
&\qquad\qquad\qquad+(e_je_k)\otimes e_i-e_i\otimes(e_je_k)\\
&\qquad\qquad\qquad-(e_ie_k)\otimes e_j+e_j\otimes(e_ie_k)\Big)\\
&\qquad\quad= a(i,j,k)\Big(e_j\otimes(e_ke_i)-e_k\otimes(e_je_i)\Big)\\
&\qquad\qquad+a(j,i,k)\Big(e_i\otimes(e_ke_j)-(e_ie_j)\otimes e_k\Big)\\
&\qquad\qquad+a(k,i,j)\Big((e_je_k)\otimes e_i-(e_ie_k)\otimes e_j\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
&X_a(i,k)(g\wedge h;h)+\mathrm{ASU}_a(i,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\
&\qquad= a(i,i,k)\Big(e_i\otimes(e_ie_k)-e_k\otimes e_i^2\Big)
\end{aligned}
$$

[1]In the first term, the variables g and h appear in this order, but in the second term they appear in the opposite order.

$$
\begin{aligned}
&+ a(k,i,k)\Big(e_i \otimes e_k^2 - e_k \otimes (e_i e_k)\Big)\\
&\qquad + \frac{a(i,i,k)}{2}\Big(e_k \otimes e_i^2 - e_i^2 \otimes e_k\Big) + \frac{a(k,i,k)}{2}\Big(e_k^2 \otimes e_i - e_i \otimes e_k^2\Big)\\
&\qquad\quad + a(k,i,k)\Big(e_k \otimes (e_i e_k) - (e_i e_k) \otimes e_k\Big)\\
&= a(i,i,k)\left(e_i \otimes (e_i e_k) - \frac{1}{2}\left(e_k \otimes e_i^2 + e_i^2 \otimes e_k\right)\right)\\
&\quad + a(k,i,k)\left(\frac{1}{2}\left(e_i \otimes e_k^2 + e_k^2 \otimes e_i\right) - (e_i e_k) \otimes e_k\right).
\end{aligned}
$$

We now compare these with $Y_a(i,j,k)$:

$$
\begin{aligned}
&Y_a(i,j,k)(\mathfrak{s}_H(g); \mathfrak{s}_H(h))\\
&\qquad = a(i,j,k)\Big(e_j \otimes (e_i e_k) - e_k \otimes (e_i e_j)\Big)\\
&\qquad\qquad + a(j,i,k)\Big(e_i \otimes (e_j e_k) - (e_i e_j) \otimes e_k\Big)\\
&\qquad\qquad\qquad + a(k,i,j)\Big((e_j e_k) \otimes e_i - (e_k e_i) \otimes e_j\Big)\\
&\qquad = X_a(i,j,k)(g \wedge h; h) + \mathrm{ASU}_a(i,j,k)(\mathfrak{s}_H(h); \mathfrak{s}_H(g))\\
&\qquad \equiv Y_a(i,j,k)(g \wedge h; h) + \mathrm{ASU}_a(i,j,k)(\mathfrak{s}_H(h); \mathfrak{s}_H(g)),
\end{aligned}
$$

and

$$
\begin{aligned}
&Y_a(i,k)(\mathfrak{s}_H(g); \mathfrak{s}_H(h))\\
&\quad = \Big(a(i,i,k)(B_{ii} \otimes e_k + e_k \otimes B_{ii} - B_{ik} \otimes e_i - e_i \otimes B_{ki})\\
&\qquad + a(k,i,k)(B_{ki} \otimes e_k + e_k \otimes B_{ik} - B_{kk} \otimes e_i - e_i \otimes B_{kk})\Big)\\
&\quad = a(i,i,k)\left(e_i \otimes (e_i e_k) - \frac{1}{2}\left(e_k \otimes e_i^2 + e_i^2 \otimes e_k\right)\right)\\
&\qquad + a(k,i,k)\left(\frac{1}{2}\left(e_i \otimes e_k^2 + e_k^2 \otimes e_i\right) - (e_i e_k) \otimes e_k\right)\\
&\quad = X_a(i,k)(g \wedge h; h) + \mathrm{ASU}_a(i,k)(\mathfrak{s}_H(h); \mathfrak{s}_H(g))\\
&\quad = Y_a(i,k)(g \wedge h; h) + \mathrm{ASU}_a(i,k)(\mathfrak{s}_H(h); \mathfrak{s}_H(g)).
\end{aligned}
$$

Therefore, we have

$$
\lambda_{a,b}(\mathfrak{s}_H(g); \mathfrak{s}_H(h)) = \lambda_{a,b}(g \wedge h; h)\frac{\mu_a(\mathfrak{s}_H(h); \mathfrak{s}_H(g))}{\mu_a(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}
$$

Since we have

$$
Y_a(mg; nh) = Y_a(m; h) + Y_a(g; n) + Y_a(g; h)
$$

for every $m, n \in M$ and $g, h \in H_{\mathrm{m}}$, we get, for each $m, n \in M$ and $g, h \in N$,

$$\begin{aligned}
&\lambda_{a,b}(m\mathfrak{s}_{H}(g); n\mathfrak{s}_{H}(h)) \\
&\quad = \lambda_{a,b}(m; \mathfrak{s}_{H}(h))\lambda_{a,b}(\mathfrak{s}_{H}(g); n)\lambda_{a,b}(\mathfrak{s}_{H}(g); \mathfrak{s}_{H}(h)) \\
&\quad = \frac{\lambda_{a,b}(m; \mathfrak{s}_{H}(h))}{\lambda_{a,b}(n; \mathfrak{s}_{H}(g))}\lambda_{a,b}(g \wedge h; h)\frac{\mu_a(\mathfrak{s}_{H}(h); \mathfrak{s}_{H}(g))}{\mu_a(\mathfrak{s}_{H}(g); \mathfrak{s}_{H}(h))} \\
&\quad = \frac{\mu_a(n\mathfrak{s}_{H}(h); (n\mathfrak{s}_{H}(h))^{-1}m\mathfrak{s}_{H}(g)n\mathfrak{s}_{H}(h))}{\mu_a(m\mathfrak{s}_{H}(g); n\mathfrak{s}_{H}(h))}.
\end{aligned}$$

This proves the cocycle identity (c). Consequently $\{\lambda_{a,b}, \mu_a\}$ is a characteristic cocycle in $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$.

ii) Suppose that $(\lambda, \mu) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. Since M is central in H_{m}, the λ-part is a bicharacter on $M \times H_{\mathrm{m}}$, so that there exists $a = \{a(i,j,k)\} \in \mathbb{R}^{\Delta}$ such that

$$\lambda(m; h) = \exp\left(2\pi \mathrm{i}\left(\sum_{i,j<k} a(i,j,k)e_{j,k}(m)e_i(h)\right)\right), \quad m \in M, h \in H_{\mathrm{m}}.$$

As $[H_{\mathrm{m}}, H_{\mathrm{m}}] = M$, for each fixed $m \in M$ the character $\lambda(m; \cdot)$ on H_{m} must vanish on M, i.e.,

$$\lambda(m; n) = 1, \quad m, n \in M.$$

Thus the restriction μ_M of the second cocycle μ to M is a coboundary. Hence, replacing μ by a cohomologous cocycle if necessary, we may and do assume that $\mu_M = 1$. Now consider the corresponding $E \in \mathrm{Xext}(H_{\mathrm{m}}, L, M, \mathbb{T})$:

$$1 \longrightarrow \mathbb{T} \longrightarrow E \underset{\overset{\longleftarrow}{\mathfrak{s}_j}}{\xrightarrow{\;j\;}} L \longrightarrow 1.$$

Redefining the cross-section $\mathfrak{s}_j$ as

$$\mathfrak{s}_j(m\mathfrak{s}_{H}(g)) = \mathfrak{s}_j(m)\mathfrak{s}_j(\mathfrak{s}_{H}(g)), \quad m \in M,\ g \in N,$$

we may and do assume that $\mu(m; g) = 1, m \in M, g \in L$. Now we compute the second cocycle μ with $m, n \in M$ and $g, h \in L$:

$$\begin{aligned}
\mu(mg; nh)\mathfrak{s}_j(mgnh) &= \mathfrak{s}_j(mg)\mathfrak{s}_j(ng) = \mathfrak{s}_j(m)\mathfrak{s}_j(g)\mathfrak{s}_j(n)\mathfrak{s}_j(h) \\
&= \mathfrak{s}_j(m)\lambda(n; g)\mathfrak{s}_j(n)\mathfrak{s}_j(g)\mathfrak{s}_j(h) \\
&= \lambda(n; g)\mu(g; h)\mathfrak{s}_j(m)\mathfrak{s}_j(n)\mathfrak{s}_j(gh) \\
&= \lambda(n; g)\mu(g; h)\mathfrak{s}_j(mngh) = \lambda(n; g)\mu(g; h)\mathfrak{s}_j(mgnh),
\end{aligned}$$

which gives

$$\mu(mg; nh) = \lambda(n; g)\mu(g; h), \quad m, n \in M, g, h \in L.$$

In particular, we have

$$\mu(g;h)=\lambda(m_0(h);g)\mu(\mathfrak{s}_H(\pi_G(g));\mathfrak{s}_H(\pi_G(h))),\quad g,h\in L,$$

where

$$m_0(h)=h\mathfrak{s}_H(\pi_G(h))^{-1}\in M.$$

Now with $g_1,g_2,g_3\in N$, we compute the coboundary:

$$\left.\begin{aligned}1&=(\partial_L\mu)(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2);\mathfrak{s}_H(g_3))\\&=\frac{\mu(\mathfrak{s}_H(g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2)\mathfrak{s}_H(g_3))}{\mu(\mathfrak{s}_H(g_1)\mathfrak{s}_H(g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2))}\\&=\frac{\mu(\mathfrak{s}_H(g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{n}_M(g_2;g_3)\mathfrak{s}_H(g_2+g_3))}{\mu(\mathfrak{n}_M(g_1;g_2)\mathfrak{s}_H(g_1+g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2))}\\&=\lambda(\mathfrak{n}_M(g_2;g_3);\mathfrak{s}_H(g_1))\frac{\mu(\mathfrak{s}_H(g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2+g_3))}{\mu(\mathfrak{s}_H(g_1+g_2);\mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1);\mathfrak{s}_H(g_2))}.\end{aligned}\right\}\tag{4.25}$$

Thus the cocycle $c_a\in \mathrm{Z}^3(N,\mathbb{T})$ given by:

$$\begin{aligned}c_a(g_1;g_2;g_3)&=\lambda(\mathfrak{n}_M(g_2;g_3);g_1)\\&=\exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k}a(i,j,k)e_{j,k}(\mathfrak{n}_M(g_2;g_3))e_i(g_1)\right)\right)\\&=\exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k}a(i,j,k)e_i(g_1)e_j(g_2)e_k(g_3)\right)\right)\end{aligned}$$

is a coboundary in $\mathrm{Z}^3(N,\mathbb{T})$. Thus we get, for every $g_1,g_2,g_3\in N$,

$$\begin{aligned}1&=\Big(\mathrm{AS}c_a\Big)(g_1,g_2,g_3)\\&=\exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k}a(i,j,k)\sum_{\sigma\in\Pi(i,j,k)}\mathrm{sign}(\sigma)e_i\big(g_{\sigma(i)}\big)e_j\big(g_{\sigma(j)}\big)e_k\big(g_{\sigma(k)}\big)\right)\right)\\&=\exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k}a(i,j,k)\det{}_{ijk}(g_1;g_2;g_3)\right)\right)\\&=\exp\left(2\pi\mathrm{i}\left(\sum_{(i,j,k)\in\Delta}(\mathrm{AS}a)(i,j,k)\det{}_{ijk}(g_1,g_2,g_3)\right)\right).\end{aligned}$$

Therefore the coefficient $a=\{a(i,j,k)\}\in\mathbb{R}^\Delta$ is asymmetric in the sense of Lemma 4.4, so that it gives the second cocycle $\mu_a\in \mathrm{Z}^2(L,\mathbb{T})$:

$$\mu_a=\exp(2\pi\mathrm{i}V_a).$$

Then the cocycle $\mu\mu_a^{-1} \in \mathrm{Z}^2(L, \mathbb{T})$ falls in the subgroup $\pi_G^*\big(\mathrm{Z}^2(N, \mathbb{T})\big) \subset \mathrm{B}^2(L, \mathbb{T})$ because

$$\begin{aligned}\mu(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h)) &= \lambda(n; \mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))\\ &= \frac{\mu_a(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h))}{\mu_a(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))\\ &= \frac{\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}{\mu_a(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}\mu_a(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h));\end{aligned}$$

$$\mu_a^{-1}\mu = \pi_G^*\circ\mathfrak{s}_H^*(\mu\mu_a^{-1}) \in \pi_G^*\Big(\mathrm{Z}^2(N, \mathbb{T})\Big).$$

Thus there exists a cochain $f \in \mathrm{C}^1(L, \mathbb{T})$ such that

$$\mu_a(g; h) = \mu(g; h)\frac{f(g)f(h)}{f(gh)}, \quad g, h \in L.$$

Since $1 = \mu(m; h) = \mu_a(m; h), m \in M, h \in L$, we have

$$f(mh) = f(m)f(h).$$

Since $(\partial_1 f)(m; h) = 1, m \in M, h \in H_{\mathrm{m}}$, we have

$$\partial f(\lambda, \mu) = (\lambda, \mu_a).$$

Next we look at one of the cocycle identities:

$$\begin{aligned}\lambda(g_1g_2; h) &= \lambda(g_1; h)\lambda(g_2; h)\frac{\mu_a(g_1; g_2)}{\mu_a(h^{-1}g_1h; h^{-1}g_2h)}\\ &= \frac{1}{\lambda(g_2 \wedge h; g_1)}\lambda(g_1; h)\lambda(g_2; h), \quad g_1, g_2 \in L, h \in H_{\mathrm{m}},\\ &= \lambda(h \wedge g_2; g_1)\lambda(g_1; h)\lambda(g_2; h)\\ &= \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k} a(i, j, k)e_i(g_1)e_{j,k}(h \wedge g_2)\right)\right)\lambda(g_1; h)\lambda(g_2; h),\end{aligned}$$

which gives the following partial coboundary condition:

$$(\partial_L \otimes \mathrm{id})\lambda = \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k} a(i, j, k)e_i \otimes (e_j \otimes e_k - e_k \otimes e_j)\right)\right).$$

Another cocycle identity:

$$\begin{aligned}\lambda(g; h_1h_2) &= \lambda(g; h_1)\lambda(h_1^{-1}gh_1; h_2), \quad g \in L, h_1, h_2 \in H_{\mathrm{m}},\\ &= \lambda(g \wedge h_1; h_2)\lambda(g; h_1)\lambda(g; h_2)\\ &= \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k} a(i, j, k)e_{j,k}(g \wedge h_1)e_i(h_2)\right)\right)\lambda(g; h_1)\lambda(g; h_2)\end{aligned}$$

gives the second partial coboundary condition:

$$(\mathrm{id}\otimes\partial_{H_{\mathrm{m}}})\lambda = \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k} a(i,j,k)(e_k\otimes e_j - e_j\otimes e_k)\otimes e_i\right)\right).$$

Setting

$$\eta_a = \exp(2\pi\mathrm{i}(Y_a)),$$

we obtain, by (4.22) and (4.23),

$$(\partial_L\otimes\mathrm{id})\lambda = (\partial_L\otimes\mathrm{id})\eta_a; \quad (\mathrm{id}\otimes\partial_{H_{\mathrm{m}}})\lambda = (\mathrm{id}\otimes\partial_{H_{\mathrm{m}}})\eta_a.$$

Therefore the cochain $\overline{\eta}_a\lambda = \chi$ is a bicharacter on $L \times H_{\mathrm{m}}$. Since $M = [H_{\mathrm{m}}, H_{\mathrm{m}}]$, the bicharacter χ vanishes on $L\times M$, i.e., $\lambda(m;g) = \eta_a(m;g), m \in M, g \in L$. Thus we get

$$\begin{aligned}1 &= \lambda(m;g)\overline{\eta}_a(m;g) = \exp(2\pi\mathrm{i}(X_a(m;g) - Y_a(m;g)))\\ &= \exp(2\pi\mathrm{i}(X_{\mathrm{AS}a}(m;g))) = \lambda_{\mathrm{AS}a}(m;g),\end{aligned}$$

which is equivalent to the following fact:

$$(\mathrm{AS}a)(i,j,k) \in \left(\frac{1}{\gcd(p_i,p_j,p_k)}\mathbb{Z}\right).$$

Thus we conclude the cocycle condition (4.9Z-a) on the parameter $\{a(i,j,k)\}$. Therefore the coefficient $a \in \mathbb{R}^{\Delta}$ satisfies the requirement for the element $(a,0) \in Z$. Therefore it follows from (i) that $(\lambda_{a,0}, \mu_a) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. Then the cocycle identity (c) for $(\lambda_{a,0}, \mu_a)$ yields that

$$\lambda(g;h) = \frac{\mu_a(h;h^{-1}gh)}{\mu_a(g;h)} = \lambda_{a,0}(g;h) = \eta_a(g;h), \quad g,h \in L.$$

Thus the bicharacter $\chi = \overline{\eta}_a\lambda$ on $L \times H_{\mathrm{m}}$ vanishes on $L \times L$. Since Lemma 4.3(i) yields for each $m \in M, h \in H_{\mathrm{m}}$ that

$$\begin{aligned}\chi(m;h) &= \lambda(m;h)\overline{\eta}_a(m;h) = \lambda_a(m;h)\overline{\eta}_a(m;h)\\ &= \lambda_{\mathrm{AS}a}(m;h) = \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k}(\mathrm{AS}a)(i,j,k)e_{j,k}(m)e_i(h)\right)\right),\end{aligned}$$

we conclude that χ is of the form:

$$\chi(g;h) = \chi_0(\pi_G(g);\pi_G(h))\exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k}(\mathrm{AS}a)(i,j,k)e_{j,k}(g)e_i(h)\right)\right)$$

for $g \in L, h \in H_{\mathrm{m}}$ where χ_0 is a bicharacter on $N \times G_{\mathrm{m}}$ and $\pi_G\colon H_{\mathrm{m}} \mapsto G_{\mathrm{m}}$ the quotient map with $M = \mathrm{Ker}(\pi_G)$. We choose $b(i,j) \in \mathbb{R}$ so that

$$\exp(2\pi\mathrm{i}(b(i,j))) = \chi_0(b_i; z_j), i \in \mathbb{N}, j \in \mathbb{N}_0.$$

Then we must have

$$1 = \chi_0(b_i; b_j) = \chi_0(b_i; p_j z_j - q_j z_0) = \exp(2\pi\mathrm{i}(b(i,j)p_j - b(i,0)q_j)),$$

so that $b(i,j) \in \mathbb{R}, i \in \mathbb{N}, j \in \mathbb{N}_0$, satisfies the following condition:

$$b(i,j)p_j \equiv b(i,0)q_j \mod \mathbb{Z}, \quad i,j \in \mathbb{N}.$$

Hence χ_0 is written in the form:

$$\chi_0(g;\tilde{h}) = \exp\left(2\pi\mathrm{i}\left(\sum_{i,j\in\mathbb{N}} b(i,j)e_{i,N}(g)e_j(\tilde{h}) + \sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(g)\tilde{e}_0(\tilde{h})\right)\right)$$

for each pair $g \in N$ and $\tilde{h} \in H_{\mathrm{m}}$, where the coefficients $b(i,j)$ satisfies the requirements:

$$b(i,j)p_j - b(i,0)q_j \in \mathbb{Z}, \quad i,j \in \mathbb{N}, \quad b(0,i) = 0, \quad i \in \mathbb{N}_0.$$

Consequently the pair (a,b) is a member of Z and we conclude that (λ,μ) is cohomologous to the characteristic cocycle $(\lambda_{a,b},\mu_a) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$.

iii) Suppose $(\lambda,\mu) = (\lambda_{a,b},\mu_a) = \partial f$ with $f \in \mathrm{C}^1(L,\mathbb{T})$. Since $\mu_M = 1$ and $\mu_a(m;g) = 1, m \in M, g \in L$, we have

$$f(mg) = f(m)f(g), \quad m \in M, g \in L,$$

so that the restriction $f|_M$ of f to M is of the form:

$$f_c(m) = \exp\left(2\pi\mathrm{i}\left(\sum_{1\le i<j} c(i,j)e_{i,j}(m)\right)\right), \quad m \in M.$$

Since M is central in H_{m}, we have for every pair $(m,g) \in M \times H_{\mathrm{m}}$

$$1 = \frac{f_c(g^{-1}mg)}{f_c(m)} = \lambda(m;g) = \exp\left(2\pi\mathrm{i}\left(\sum_{i,j<k} a(i,j,k)e_{j,k}(m)e_i(g)\right)\right),$$

which yields the integrality condition on a:

$$a(i,j,k) \in \mathbb{Z} \quad \text{for every } (i,j,k) \in \Delta.$$

Hence $Y_a(i,j,k)$ and $X_a(i,j,k)$ are both integer valued, so that

$$\lambda_a(m;h) = 1, \quad m \in M, h \in H_{\mathrm{m}}.$$

Since $\chi = 1$ on $L \times L$, for every $g, h \in L$ we have

$$\begin{aligned}
1 &= \lambda_{0,b}(g;h) = \lambda_a(m_0(g);h)\lambda_{0,b}(\mathfrak{s}_H(g);h) = \lambda(g;h)\\
&= \frac{f(h^{-1}gh)}{f(g)} = f_c(g \wedge h);\\
c(i,j) &\in \left(\frac{1}{p_i p_j}\mathbb{Z}\right), \quad i,j \in \mathbb{N}.
\end{aligned}$$

This computation also shows that

$$\lambda_{0,b}(g;h) = f_c(g \wedge h), \quad g \in L, h \in H_{\mathrm{m}}.$$

Furthermore, we have for each $m, n \in M$ and $g, h \in L$

$$\mu_a(mg;nh) = \lambda_{a,b}(n;g)\mu_a(g;h) = \mu_a(g;h),$$

so that μ_a is of the form: $\mu_a = \pi_G^*(\tilde{\mu})$ with

$$\tilde{\mu}_a(g;h) = \exp(2\pi\mathrm{i}(U_a(g;h))), \quad g,h \in N.$$

Since $\lambda_a(\mathfrak{n}_M(g_2;g_3);g_1) = 1$, $g_1, g_2, g_3 \in H_{\mathrm{m}}$, we have $\tilde{\mu}_a \in \mathrm{Z}^2(N,\mathbb{T})$ by (4.25). We first compute for each $g, h \in L$:

$$(\mathrm{AS}\mu_a)(g;h) = \frac{f(g)f(h)}{f(gh)}\frac{f(hg)}{f(g)f(h)} = \frac{f(hgh^{-1}h)}{f(gh)} = f_c(h \wedge g) = 1.$$

Since $\mathrm{AS}U_a(i,j,k)$ is also integer valued, we have

$$\begin{aligned}
\mathrm{AS}\mu_a &= \exp\left(2\pi\mathrm{i}\left(\sum_{i<k}\mathrm{AS}U_a(i,k)\right)\right)\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i<k}\left(\frac{a(i,i,k)}{2}\left(e_i^2\otimes e_k - e_k\otimes e_i^2\right)\right)\right)\right)\\
&\qquad\times\exp\left(2\pi\mathrm{i}\left(\sum_{i<k}\frac{a(k,i,k)}{2}\left(e_i\otimes e_k^2 - e_k^2\otimes e_i\right)\right)\right)\\
&= 1.
\end{aligned}$$

Thus we get

$$a(i,i,k), \quad a(k,i,k) \in 2\mathbb{Z}, \quad \text{and} \quad U_a(i,k) \equiv 0 \mod \mathbb{Z}.$$

Consequently, $\tilde{\mu}_a$ is a coboundary as a member of $\mathrm{Z}^2(N, \mathbb{T})$. Hence there exists a cochain $\tilde{f} \in \mathrm{C}^1(N, \mathbb{T})$ such that

$$\frac{f(g)f(h)}{f(gh)} = \mu_a(g; h) = \tilde{\mu}_a(\pi_G(g); \pi_G(h)) = \frac{\tilde{f}(\pi_G(g))\tilde{f}(\pi_G(h))}{\tilde{f}(\pi_G(gh))}.$$

Thus f is of the form:

$$f(g) = f_c(m_0(g))f(\mathfrak{s}_H(g)) = \chi(g)\tilde{f}(\pi_G(g)), \quad g \in L;$$
$$f_c(m) = \chi(m), \quad m \in M.$$

where $\chi \in \mathrm{Hom}(L, \mathbb{T})$. Since

$$L/[L, L] \cong M/PMP \oplus N,$$

the homomorphism χ is of the form:

$$\chi(g) = \exp\left(2\pi \mathrm{i}\left(\sum_{j<k} c(j,k)e_{j,k}(g) + \sum_{k\in\mathbb{N}_0} c(k)\tilde{e}_k(g)\right)\right), \quad g \in L,$$

where

$$c(i,j) \in \left(\frac{1}{p_i p_j}\mathbb{Z}\right), \ i < j, \quad \text{and} \quad c(k) \in \mathbb{R}.$$

Since Y_a is integer valued, the λ-part becomes the following:

$$\begin{aligned}
\lambda(g; h) &= \exp\left(2\pi \mathrm{i}\left(\sum_{j\in\mathbb{N}, k\in\mathbb{N}_0} b(j,k)e_{j,N}(g)\tilde{e}_k(h)\right)\right), \quad g \in N, \ h \in H_{\mathrm{m}}, \\
&= \frac{f(h^{-1}gh)}{f(g)} = \frac{f((g \wedge h)g)}{f(g)} = f_c(g \wedge h) \\
&= \exp\left(2\pi \mathrm{i}\left(\sum_{1\le j<k} c(j,k)e_{j,k}(g \wedge h)\right)\right) \\
&= \exp\left(2\pi \mathrm{i}\left(\sum_{1\le j<k} c(j,k)\Big(e_j(g)e_k(h) - e_k(g)e_j(h)\Big)\right)\right) \\
&= \exp\left(2\pi \mathrm{i}\left(\sum_{1\le j<k} c(j,k)\Big(p_j e_{j,N}(g)e_k(h) - p_k e_{k,N}(g)e_j(h)\Big)\right)\right).
\end{aligned}$$

Hence we conclude that for $j < k$ and $i \in \mathbb{N}_0$

$$b(i, 0) \in \mathbb{Z}, \quad b(i,i) \in \mathbb{Z}, \ i \in \mathbb{N};$$
$$b(j,k) \equiv c(j,k)p_j, \ b(k,j) \equiv -c(j,k)p_k \mod \mathbb{Z}, \quad j < k.$$

Thus we have for $i < j$

$$b(i,j) = c(i,j)p_i + m_{i,j} \quad \text{with some } m_{i,j} \in \mathbb{Z};$$
$$b(j,i) = -c(i,j)p_j + m_{j,i} \quad \text{with some } m_{j,i} \in \mathbb{Z};$$
$$\frac{b(i,j)}{p_i} + \frac{b(j,i)}{p_j} = \frac{m_{i,j}}{p_i} + \frac{m_{j,i}}{p_j} \in \left(\frac{1}{p_i}\mathbb{Z}\right) + \left(\frac{1}{p_j}\mathbb{Z}\right) = \left(\frac{1}{\mathrm{lcm}(p_i,p_j)}\mathbb{Z}\right).$$

Conversely suppose $(a,b) \in \mathrm{B}$, i.e.,

$$a(i,j,k) \in \mathbb{Z} \quad \text{for} \quad i<j<k;$$
$$a(i,i,k),\ a(k,i,k) \in 2\mathbb{Z} \quad \text{for} \quad i<k,$$

and

$$\frac{b(i,j)}{p_i} + \frac{b(j,i)}{p_j} \in \left(\frac{1}{\mathrm{lcm}(p_i,p_j)}\mathbb{Z}\right) \quad \text{and} \quad b(i,i) \in \mathbb{Z};$$
$$b(i,0) \in \mathbb{Z}, \quad i \in \mathbb{N}.$$

So we can write

$$\frac{b(i,j)}{p_i} + \frac{b(j,i)}{p_j} = \frac{m_{i,j}}{p_i} + \frac{m_{j,i}}{p_j} \quad \text{with} \quad m_{i,j}, m_{j,i} \in \mathbb{Z}.$$

Set

$$c(i,j) = \frac{b(i,j)}{p_i} - \frac{m_{i,j}}{p_i} \quad \text{for } i<j, \qquad c(i,i) = b(i,i),$$

so that

$$\frac{b(j,i)}{p_j} = -c(i,j) + \frac{m_{j,i}}{p_j}.$$

Then we have

$$\begin{aligned}
&\sum_{i,j\in\mathbb{N}} b(i,j)e_{i,N}(g)e_j(h) \\
&\qquad \equiv \sum_{i<j} c(i,j)\Big(p_i e_{i,N}(g)e_j(h) - p_j e_{j,N}(g)e_i(h)\Big) \mod \mathbb{Z}, \\
&\qquad = \sum_{i<j} c(i,j)e_{i,j}(g\wedge h).
\end{aligned}$$

Thus with

$$f_c(g) = \exp\left(2\pi\mathrm{i}\left(\sum_{1\le i<j} c(i,j)e_{i,j}(g)\right)\right), \quad g \in L,$$

we have

$$\exp\left(2\pi\mathrm{i}\left(\sum_{i,j} b(i,j)e_{i,N}(g)e_j(h)\right)\right) = \frac{f_c(h^{-1}gh)}{f_c(g)} = \partial_1 f_c(g;h),$$

where $e_{i,N}(g)$ means $e_{i,N}\circ\pi_G$. We then compute the coboundary of f_c:

$$\begin{aligned}\left(\partial_L f_c\right)(g;h) &= \frac{f_c(g)f_c(h)}{f_c(gh)}, \quad g,h\in L,\\ &= \exp\left(2\pi\mathrm{i}\left(\sum_{i<j}c(i,j)(e_{i,j}(g)+e_{i,j}(h)-e_{i,j}(gh))\right)\right)\\ &= \exp\left(-2\pi\mathrm{i}\sum_{i<j}c(i,j)e_i(g)e_j(h)\right)=1,\end{aligned}$$

because $e_i(g)\in p_i\mathbb{Z}$ and $e_j(h)\in p_j\mathbb{Z}$ if $g,h\in L$ and

$$p_i c(i,j)p_j = b(i,j)p_j - m_{i,j}p_j \equiv b(i,0)q_j \equiv 0 \mod \mathbb{Z}.$$

As $a(i,j,k)\in\mathbb{Z}$ for every triplet $(i,j,k)\in\Delta$, we get trivially

$$\lambda_{a,0}=1, \quad \tilde{\mu}_a=\mathfrak{s}_H^*\mu_a\in \mathrm{Z}^2(N,\mathbb{T}), \quad \text{and} \quad \mu_a=\pi_G^*(\tilde{\mu}_a).$$

Since $\partial_N U_a(i,j,k), i<j<k$, is integer valued, the cochain

$$\tilde{\mu}_a^{ijk}=\exp(2\pi\mathrm{i}(U_a(i,j,k)))$$

belongs to $\mathrm{Z}^2(N,\mathbb{T})$. Since $\mathrm{AS}U_a(i,j,k)$ is integer valued by (4.24), $\mathrm{AS}\tilde{\mu}_a^{ijk}=1$ and therefore $\tilde{\mu}_a^{ijk}\in\mathrm{B}^2(N,\mathbb{T})$. In view of the fact that

$$\tilde{\mu}_a^{ik}=\exp(2\pi\mathrm{i}(U_a(i,k)))=1, \quad i<k,$$

we conclude that $\tilde{\mu}_a\in\mathrm{B}^2(N,\mathbb{T})$. Thus there exists a cochain $\tilde{f}\in\mathrm{C}^1(N,\mathbb{T})$ such that

$$\tilde{\mu}_a=\partial_N\tilde{f}.$$

Define a cochain $f\in\mathrm{C}^1(L,\mathbb{T})$ by

$$f=(\pi_G^*\tilde{f})f_c.$$

Then we get for each pair $g\in L, h\in H_{\mathrm{m}}$

$$\begin{aligned}(\partial_1 f)(g;h) &= \frac{f(h^{-1}gh)}{f(g)} = \frac{\tilde{f}(\pi_G(h^{-1}gh))f_c(h^{-1}gh)}{\tilde{f}(\pi_G(g))f_c(g)}\\ &= \frac{f_c(h^{-1}gh)}{f_c(g)} = \lambda_{a,b}(g;h);\\ (\partial_2 f)(g;h) &= \frac{\tilde{f}(\pi_G(g))f_c(g)\tilde{f}(\pi_G(h))f_c(h)}{\tilde{f}(\pi_G(gh))f_c(gh)}, \quad g,h\in L,\\ &= \partial_L f_c(g;h)\left(\partial_N\tilde{f}\right)(\pi_G(g);\pi_G(h))\\ &= \tilde{\mu}_a(\pi_G(g);\pi_G(h)) = \mu_a(g;h).\end{aligned}$$

Therefore we conclude

$$\partial f=\{\lambda_{a,b},\mu_a\}\in\mathrm{B}(H_{\mathrm{m}},L,M,\mathbb{T}).$$

This completes the proof. ♡

Lemma 4.6. *The cocycle λ_b corresponding to $b \in Z_b$ does not depend on the M-component, i.e.,*

$$\lambda_b(mg; n\tilde{h}) = \lambda_b(g; \tilde{h}), \quad m, n \in M, g \in L, \tilde{h} \in H_{\mathrm{m}}.$$

we will view λ_b as a bicharacter on $N \times G_{\mathrm{m}}$ rather than on $L \times H_{\mathrm{m}}$.

i) *For $i \in \mathbb{Z}$, set*

$$\mathrm{Z}_b(i,i) = \big\{z = (x,u) \in \mathbb{R}^2 : p_i x - q_i u \in \mathbb{Z}\big\}, \quad \mathrm{B}_b(i,i) = \mathbb{Z} \oplus \mathbb{Z}.$$

The bicharacter $\lambda_z^{i,i}$ on $N \times G_{\mathrm{m}}$ determined by:

$$\lambda_z^{i,i}(g;h) = \exp(2\pi \mathrm{i}(x e_{i,N}(g)\tilde{e}_i(h) + u e_{i,N}(g)\tilde{e}_0(h))), g \in N, h \in G_{\mathrm{m}},$$

gives a characteristic cocycle of $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. It is a coboundary if and only if z is in $\mathrm{B}_b(i,i)$. The corresponding cohomology class $[\lambda_z^{i,i}] \in \Lambda_b(i,i)$ is given by:

$$\big[\lambda_z^{i,i}\big] = \Big([p_i x - q_i u]_{\gcd(p_i,q_i)}, [-v_i x + u_i u]_{\mathbb{Z}}\Big) \in \mathbb{Z}_{\gcd(p_i,q_i)} \oplus (\mathbb{R}/\mathbb{Z}),$$

where the integers u_i, v_i are determined by:

$$p_i u_i - q_i v_i = \gcd(p_i, q_i)$$

through the Euclid algorithm.

ii) *Fix a pair $i, j \in \mathbb{N}$ of indices and set*

$$\begin{gathered}\mathrm{Z}_b(i,j) = \big\{(x,u,y,v) \in \mathbb{R}^4 : p_j x - q_j u \in \mathbb{Z}, p_i y - q_i v \in \mathbb{Z}\big\};\\ \mathrm{B}_b(i,j) = \{(x,u,y,v) \in \mathrm{Z}_b(i,j) : p_j x + p_i y \in \gcd(p_i,p_j)\mathbb{Z}, u, v \in \mathbb{Z}\}.\end{gathered}$$

To each element $z = (x,u,y,v) \in \mathrm{Z}_b(i,j)$, there corresponds a bicharacter λ_z on $N \times G_{\mathrm{m}}$ determined by:

$$\begin{aligned}\lambda_z^{i,j}(g;h) = {}&\exp(2\pi\mathrm{i}(x e_{i,N}(g)\tilde{e}_j(h) + y e_{j,N}(g)\tilde{e}_i(h)))\\ &\times \exp(2\pi\mathrm{i}(u e_{i,N}(g)\tilde{e}_0(h) + v e_{j,N}(g)\tilde{e}_0(h))),\end{aligned} \quad g \in N, h \in G_{\mathrm{m}},$$

which is a characteristic cocycle in $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. It is a coboundary if and only if $z \in \mathrm{B}_b(i,j)$. The cohomology class $[\lambda_z^{i,j}] \in \Lambda_b(i,j)$ of λ_z corresponds to the parameter class:

$$[z] = \begin{pmatrix} [m_{i,j}(x r_{j,i} + y r_{i,j}) - n_{i,j}(u s_{j,i} + v s_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(x r_{j,i} + y r_{i,j}) + x_{i,j}(u s_{j,i} + v s_{i,j})]_{\mathbb{Z}} \\ [-u w_{i,j} + v w_{j,i}]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} \left(\dfrac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix},$$

where $D(i,j), \cdots, w_{j,i}$ are given in (4.17) *of* Theorem 4.2.

Proof. i) Set

$$D_i = \gcd(p_i, q_i), \quad r_i = \frac{p_i}{D_i}, \quad s_i = \frac{q_i}{D_i},$$

and choose integers $u_i, v_i \in \mathbb{Z}$ so that

$$r_i u_i - s_i v_i = 1,$$

where such a pair $(u_i, v_i) \in \mathbb{Z}^2$ can be determined through the Euclid algorithm. Next we set

$$\begin{gathered} e_1 = (1,0), \quad e_2 = (0,1); \\ f_1 = u_i e_1 + v_i e_2, \quad e_1 = r_i f_1 - v_i f_2, \\ f_2 = s_i e_1 + r_i e_2, \quad e_2 = -s_i f_1 + u_i f_2. \end{gathered}$$

Then $\mathrm{Z}_b(i,i)$ is given by the following:

$$\mathrm{Z}_b(i,i) = \left(\frac{1}{D_i}\mathbb{Z}\right) f_1 + \mathbb{R} f_2,$$

and

$$\mathrm{B}_b(i,i) = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z}f_1 + \mathbb{Z}f_2,$$

so that

$$\Lambda_b(i,i) = \mathrm{Z}_b(i,i)/\mathrm{B}_b(i,i) \cong \left(\frac{1}{D_i}\mathbb{Z}\Big/\,\mathbb{Z}\right)\dot{f}_1 \oplus (\mathbb{R}/\mathbb{Z})\dot{f}_2,$$

where the dotted elements indicate the corresponding elements in the quotient group $\Lambda_b(i,i)$. Now we chase the parameter:

$$\begin{aligned} z &= xe_1 + ue_2 = x(r_i f_1 - v_i f_2) + u(-s_i f_1 + u_i f_2) \\ &= (r_i x - s_i u) f_1 + (-v_i x + u_i u) f_2; \\ \dot{z} &= [r_i x - s_i u]_{\mathbb{Z}} \dot{f}_1 + [-v_i x + u_i u]_{\mathbb{Z}} \dot{f}_2. \end{aligned}$$

$$\lambda_z^{i,i}(g; \tilde{h}) = \exp\Big(2\pi \mathrm{i}\Big(\Big(x e_{i,N}(g) e_i(\tilde{h}) + u e_{i,N}(g) e_0(\tilde{h})\Big)\Big)\Big)$$

for each pair $g \in N$ and $\tilde{h} \in G_{\mathrm{m}}$.

ii) First we fix the standard basis $\{e_1, \cdots, e_4\}$ of $\mathbb{R}^4$ and set

$$g_0 = r_{i,j} e_1 - r_{j,i} e_3, \quad g_1 = u_{j,i} e_1 + u_{i,j} e_3,$$

where we choose $u_{i,j}, u_{j,i} \in \mathbb{Z}$ so that

$$r_{i,j} u_{i,j} + r_{j,i} u_{j,i} = 1.$$

Since

$$e_1 = u_{i,j}g_0 + r_{j,i}g_1, \quad e_2 = -u_{j,i}g_0 + r_{i,j}g_1,$$

we have

$$\mathbb{Z}e_1 + \mathbb{Z}e_3 = \mathbb{Z}g_0 + \mathbb{Z}g_1.$$

Also we have

$$\mathrm{B}_b(i,j) + \mathbb{R}g_0 = \mathbb{R}g_0 + \mathbb{Z}g_1 + \mathbb{Z}e_2 + \mathbb{Z}e_4.$$

Consider an integer 3×4-matrix:

$$T = \begin{pmatrix} m_{i,j}r_{j,i} & -n_{i,j}s_{j,i} & m_{i,j}r_{i,j} & -n_{i,j}s_{i,j} \\ y_{i,j}r_{j,i} & x_{i,j}s_{j,i} & y_{i,j}r_{i,j} & x_{i,j}s_{i,j} \\ 0 & -w_{i,j} & 0 & w_{j,i} \end{pmatrix}.$$

We claim that

$$T(\mathrm{Z}_b(i,j) + \mathbb{R}g_0) = \left(\frac{1}{D(i,j)}\mathbb{Z}\right) \oplus \mathbb{R} \oplus \mathbb{R}.$$

To prove the claim, for each vector

$$z = xe_1 + ue_2 + ye_3 + ve_4 \in \mathbb{R}^4,$$

we simply compute,

$$Tg_0 = 0,$$

$$\begin{aligned} Tz &= \begin{pmatrix} m_{i,j}r_{j,i} & -n_{i,j}s_{j,i} & m_{i,j}r_{i,j} & -n_{i,j}s_{i,j} \\ y_{i,j}r_{j,i} & x_{i,j}s_{j,i} & y_{i,j}r_{i,j} & x_{i,j}s_{i,j} \\ 0 & -w_{i,j} & 0 & w_{j,i} \end{pmatrix} \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix} \\ &= \begin{pmatrix} m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \\ y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j}) \\ -uw_{i,j} + vw_{j,i} \end{pmatrix}. \end{aligned}$$

Suppose

$$\frac{k}{D(i,j)} = m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \in \left(\frac{1}{D(i,j)}\right)\mathbb{Z}.$$

Then we have

$$\begin{aligned} k &= (m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}))D(i,j) \\ &= (xp_j - uq_j) + (yp_i - vq_i) \\ &= ((x + tr_{i,j})p_j - uq_j) + ((y - tr_{j,i})p_i - vq_i). \end{aligned}$$

A choice of $t \in \mathbb{R}$, such that $(x + tr_{i,j})p_j - uq_j$ is an integer, yields the integrality of the other term $(y - tr_{j,i})p_i - vq_i$, so that

$$z + tg_0 \in \mathrm{Z}_b(i,j).$$

Now we prove that

$$T^{-1}\mathbb{Z}^3 = \mathrm{B}_b(i,j) + \mathbb{R}g_0.$$

Since T is a matrix with integer coefficients and the generators g_1, e_2, e_4 are all integer vectors, we have $T(\mathrm{B}_b(i,j)) \subset \mathbb{Z}^3$. Conversely suppose that $Tz \in \mathbb{Z}^3$. Then we have

$$\begin{gathered}
k = m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \in \mathbb{Z}, \\
\ell = y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j}) \in \mathbb{Z}, \\
m = -uw_{i,j} + vw_{j,i} \in \mathbb{Z}.
\end{gathered}$$

Hence we get

$$\begin{gathered}
xr_{j,i} + yr_{i,j} = x_{i,j}k + n_{i,j}\ell \in \mathbb{Z}, \quad n = us_{j,i} + vs_{i,j} = -y_{i,j}k + m_{i,j}\ell \in \mathbb{Z}, \\
u = nw_{j,i} - ms_{i,j} \in \mathbb{Z}, \quad v = nw_{i,j} + ms_{j,i} \in \mathbb{Z}, \\
xp_j + yp_i = (xr_{j,i} + yr_{i,j})D_{i,j} \in D_{i,j}\mathbb{Z}.
\end{gathered}$$

Therefore $z \in \mathrm{B}_b(i,j) + \mathbb{R}g_0$.

Consequently, we conclude

$$\Lambda_b(i,j) \cong \mathrm{Z}_b(i,j)/\mathrm{B}_b(i,j) \cong \left(\left(\frac{1}{D(i,j)}\mathbb{Z} \right) \Big/ \mathbb{Z} \right) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}),$$

in the sense that the cohomology class $\left[\lambda_z^{i,j}\right] \in \Lambda_b(i,j)$ corresponds to the following:

$$[z] = \begin{pmatrix} [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [-uw_{i,j} + vw_{j,i}]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} \left(\frac{1}{D(i,j)}\mathbb{Z}\right) \Big/ \mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix}.$$

For each $i,j \in \mathbb{N}$, define maps $\pi_i^i : \Lambda_b(i,i) \mapsto \mathbb{R}/\mathbb{Z}$, $\pi_{i,j}^i : \Lambda_b(i,j) \mapsto \mathbb{R}/\mathbb{Z}$, $\pi_{i,j}^j : \Lambda_b(i,j) \mapsto \mathbb{R}/\mathbb{Z}$ and $\pi_{ij} : \Lambda_b(i,j) \mapsto \left(\frac{1}{D(i,j)}\mathbb{Z}\right) \Big/ \mathbb{Z}$ by the following:

$$\begin{gathered}
\pi_i^i([\lambda_z^{i,i}]) = [u]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}, \\
\pi_{ii}\left(\left[\lambda_z^{i,i}\right]\right) = [xr_i - us_i]_{\mathbb{Z}} \in \left(\frac{1}{D_i}\mathbb{Z}\right) \Big/ \mathbb{Z}
\end{gathered}$$

for each $z = (x,u) \in \mathrm{Z}_b(i,i)$, and

$$\begin{gathered}
\pi_{i,j}^i([\lambda_z^{i,j}]) = [u]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}, \quad \pi_{i,j}^j([\lambda_z^{i,j}]) = [v]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}, \\
\pi_{ij}\left(\left[\lambda_z^{i,j}\right]\right) = [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \in \left(\frac{1}{D(i,j)}\mathbb{Z}\right) \Big/ \mathbb{Z}
\end{gathered}$$

for each $z = (x, u, y, v) \in \mathrm{Z}_b(i, j)$. The above maps $\pi^i_{i,j}$ and $\pi^j_{i,j}$ are both well-defined because the coboundary condition on z implies the integrality of u and v.

Let Λ_b be the set of all those

$$\lambda_b = \{\lambda_b(i,i), \lambda_b(i,j)\} \in \prod_{i\in\mathbb{N}} \Lambda_b(i,i) \times \prod_{\substack{i<j\\ i,j\in\mathbb{N}}} \Lambda_b(i,j)$$

such that

$$\pi^i_i(\lambda_b(i,i)) = \pi^i_{i,j}(\lambda_b(i,j)) = \pi^i_{ki}(\lambda_b(k,i)) \quad \text{for all} \;\; i,j,k \in \mathbb{N}.$$

Finally we have

$$\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) = \Lambda_a \oplus \Lambda_b.$$

This completes the proof. ♡

Remark 4.6. The direct sum homomorphism $\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}$ is a homomorphism of $\Lambda_a(i,j)$ onto the direct sum group:

$$\Lambda_b(i,j) \xrightarrow{\pi_{ij}\oplus\pi^i_{i,j}\oplus\pi^j_{i,j}} \left(\left(\frac{1}{D(i,j)}\mathbb{Z}\right)\Big/\,\mathbb{Z}\right) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}).$$

i) By multiplying D_i to $\pi_{ii}(\lambda_z)$ we get

$$D_i\pi_{ii}([\lambda_z]) = [xp_i - uq_i]_{D_i\mathbb{Z}} \in \mathbb{Z}/(D_i\mathbb{Z});$$

Similarly, we have

$$D(i,j)\pi_{ij}(\lambda_z) = [(xp_j + yp_i) - (uq_j + vq_i)]_{D(i,j)} \in \mathbb{Z}/(D(i,j)\mathbb{Z}).$$

ii) The kernel of $\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}$ is given by the following:

$$\mathrm{Ker}\Big(\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}\Big) = \begin{pmatrix} \{0\} \\ \left(\dfrac{1}{m_{i,j}}\mathbb{Z}\right)\Big/\,\mathbb{Z} \\ \{0\} \end{pmatrix}.$$

At the parameter level, the kernel is described as follows:

$$[\lambda_z] \in \mathrm{Ker}\Big(\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}\Big) \Leftrightarrow xp_j + yp_i \in D(i,j)\mathbb{Z}, u, v \in \mathbb{Z}.$$

§5. The Reduced Modified HJR-Sequence.

We are now going to investigate the reduced modified HJR-exact sequence:

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
\mathrm{H}^2(H,\mathbb{T}) \arrow[d, "\mathrm{Res}"'] \arrow[rr, equal] & & \mathrm{H}^2(H,\mathbb{T}) \arrow[d, "\mathrm{res}"'] \\
\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) \arrow[d, "\delta"'] \arrow[rr, "\mathrm{res}"] & & \Lambda(H, M, \mathbb{T}) \arrow[d, "\delta_{\mathrm{HJR}}"'] \\
\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) \arrow[d, "\mathrm{Inf}"'] \arrow[rr, "\partial_{Q_{\mathrm{m}}}"] & & \mathrm{H}^3(G,\mathbb{T}) \arrow[d, "\mathrm{inf}"'] \\
\mathrm{H}^3(H,\mathbb{T}) \arrow[rr, equal] & & \mathrm{H}^3(H,\mathbb{T})
\end{tikzcd} \tag{5.1}$$

We refer to [KtT3: page 116] for detail. So we first discuss the second cohomology group $\mathrm{Z}^2(H,\mathbb{T})$ and the restriction map Res. Each second cocycle $\mu \in \mathrm{Z}^2(H,\mathbb{T})$ gives rise to a group extension equipped with a cross-section:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r] & E \arrow[r, "j"] & H \arrow[l, bend left, "\mathfrak{s}_j"] \arrow[r] & 1
\end{tikzcd}$$

such that

$$\mathfrak{s}_j(g)\mathfrak{s}_j(h) = \mu(g;h)\mathfrak{s}_j(gh), \quad g,h \in H.$$

With

$$\lambda_\mu(g;h) = \frac{\mu(h;h^{-1}gh)}{\mu(g;h)}, \quad g,h \in H,$$

we obtain a characteristic cocycle $(\lambda_\mu, \mu) \in \mathrm{Z}(H,H,\mathbb{T})$. This corresponds to the case that $P = 1$ in the previous section. So we set

$$\begin{aligned}
\mathrm{Z}^2 &= \left\{ a \in \mathbb{R}^{\mathbb{N}^3} : a(i,j,k) = 0 \text{ if } j \geq k, (\mathrm{AS}a)(i,j,k) \in \mathbb{Z} \right\}; \\
\mathrm{B}^2 &= \left\{ a \in \mathrm{Z}^2 : a(i,j,k) \in \mathbb{Z}, a(i,i,k), a(k,i,k) \in 2\mathbb{Z} \right\}.
\end{aligned} \tag{5.2}$$

Theorem 5.1. i) *Each element $a \in \mathrm{Z}^2$ gives rise to a cocycle $\mu_a \in \mathrm{Z}^2(H,\mathbb{T})$:*

$$\mu_a = \exp(2\pi \mathrm{i} V_a) \in \mathrm{Z}^2(H,\mathbb{T}), \quad a \in \mathrm{Z}^2, \tag{5.3}$$

and the following diagram describes the second cohomology $\mathrm{H}^2(H,\mathbb{T})$:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{B}^2 \arrow[r] \arrow[d] & a \in \mathrm{Z}^2 \arrow[r] \arrow[d] & {[a] \in \mathrm{H}^2} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{B}^2(H,\mathbb{T}) \arrow[r] & \mu_a \in \mathrm{Z}^2(H,\mathbb{T}) \arrow[r] & {[\lambda_a] \in \mathrm{H}^2(H,\mathbb{T})} \arrow[r] & 1
\end{tikzcd}$$

More precisely, with

$$\left.\begin{aligned}\mathrm{Z}^2(i,j,k)=\{(x,y,z)\in\mathbb{R}^3: x-y+z\in\mathbb{Z}\},\quad \mathrm{B}^2(i,j,k)=\mathbb{Z}^3,\\ \mathrm{Z}^2(i,k)=\mathbb{R}^2,\quad \mathrm{B}^2(i,k)=(2\mathbb{Z})^2,\\ \mathrm{H}^2(i,j,k)=\mathrm{Z}^2(i,j,k)/\mathrm{B}^2(i,j,k),\quad \mathrm{H}^2(i,k)=\mathrm{Z}^2(i,k)/\mathrm{B}^2(i,k)\end{aligned}\right\}\tag{5.4}$$

for each triplet $i<j<k$ (resp. pair $i<k$) and

$$a(i,j,k)=x,\quad a(j,i,k)=y,\quad a(k,i,j)=z,$$
$$(\text{resp.}\quad a(i,i,k)=x,\quad a(k,i,k)=y),$$

we set

$$\mu_a^{ijk}=\exp(2\pi\mathrm{i}(V_a(i,j,k)))\in\mathrm{Z}^2(H,\mathbb{T});$$
$$\mu_a^{ik}=\exp(2\pi\mathrm{i}(V_a(i,k)))\in\mathrm{Z}^2(H,\mathbb{T}).$$

Then we have

$$\mathrm{Z}^2(H,\mathbb{T})=\prod_{i<j<k}\mathrm{Z}^2(i,j,k)\times\prod_{i<k}\mathrm{Z}^2(i,k),$$
$$\mathrm{B}^2(H,\mathbb{T})=\prod_{i<j<k}\mathrm{B}^2(i,j,k)\times\prod_{i<k}\mathrm{B}^2(i,k),$$
$$\mu_a=\left(\prod_{i<j<k}\mu_a^{ijk}\right)\left(\prod_{i<k}\mu_a^{ik}\right)\in\mathrm{Z}^2(H,\mathbb{T}),$$
$$\mathrm{H}^2(H,\mathbb{T})\cong\prod_{i<j<k}\mathrm{H}^2(i,j,k)\times\prod_{i<k}\mathrm{H}^2(i,k),$$
$$[\mu_a]=\big([\mu_a^{ijk}],[\mu_a^{ik}]:i<j<k\text{ and }i<k\big)\in\mathrm{H}^2(H,\mathbb{T}).$$

Each $\mathrm{H}^2(i,j,k), i<j<k$, *(resp.* $\mathrm{H}^2(i,k), i<k$*), is given by:*

$$\mathrm{H}^2(i,j,k)\cong(\mathbb{R}/\mathbb{Z})\oplus(\mathbb{R}/\mathbb{Z}),$$
$$\big(\text{resp.}\quad \mathrm{H}^2(i,k)\cong(\mathbb{R}/2\mathbb{Z})\oplus(\mathbb{R}/2\mathbb{Z})\big).$$

Proof. Most of the claims have been proved already except the claim for the structure of $\mathrm{H}^2(i,j,k)$. To prove the assertion on $\mathrm{H}^2(i,j,k)$, it is convenient to introduce a matrix $A\in\mathrm{SL}(3,\mathbb{Z})$:

$$A=\begin{pmatrix}1&-1&1\\0&1&0\\0&0&1\end{pmatrix}\in\mathrm{SL}(3,\mathbb{Z}),\quad A^{-1}=\begin{pmatrix}1&1&-1\\0&1&0\\0&0&1\end{pmatrix}.$$

We then observe that

$$A\mathrm{Z}^2(i,j,k)=(\mathbb{Z}\oplus\mathbb{R}\oplus\mathbb{R}),\quad A\mathrm{B}^2=\mathbb{Z}^3,$$

and conclude

$$\mathrm{H}^2(i,j,k)\cong\{0\}\oplus(\mathbb{R}/\mathbb{Z})\oplus(\mathbb{R}/\mathbb{Z}).$$

This completes the proof. ♡

Theorem 5.2. i) *Each second cocycle* $\mu_a \in \mathrm{Z}^2(H, \mathbb{T})$, $a \in \mathrm{Z}^2$, *gives the corresponding characteristic cocycle*:

$$\mathrm{Res}(\mu_a) = (\lambda_a, \mu_a) = \pi_{\mathrm{m}}^*(\lambda_a|_{L\times H_{\mathrm{m}}}, \mu_a|_L) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T}).$$

The image $\mathrm{Res}\big(\mathrm{Z}^2(H, \mathbb{T})\big)$ *is therefore given by:*

$$\mathrm{Res}\big(\mathrm{Z}^2(H, \mathbb{T})\big) = \{(\lambda_a, \mu_a) : a \in \mathrm{Z}_a, (\mathrm{AS}a)(i, j, k) \in \mathbb{Z}, i < j < k\}.$$

The (i, j, k)*-component* $\mathrm{Res}(i, j, k)$ *of the restriction map* Res *gives rise to the following commutative diagram of short exact sequences*:

$$\begin{tikzcd}
1 \arrow[d] & & 1 \arrow[d] \\
\mathrm{B}^2(i,j,k) = \mathbb{Z}^3 \arrow[d] \arrow[rr, "X_a(i,j,k)\longrightarrow X_a(i,j,k)"] & & \mathrm{B}_a(i,j,k) = \mathbb{Z}^3 \arrow[d] \\
\mathrm{Z}^2(i,j,k) = A^{-1}(\mathbb{Z}\oplus\mathbb{R}^2) \arrow[d] \arrow[rr, "X_a(i,j,k)\longrightarrow X_a(i,j,k)"] & & \mathrm{Z}_a(i,j,k) = A^{-1}\left(\frac{1}{D}\mathbb{Z}\oplus\mathbb{R}^2\right) \arrow[d] \\
\mathrm{H}^2(i,j,k) = \{0\}\oplus\mathbb{T}^2 \arrow[d] \arrow[rr, "\mathrm{Res}(i,j,k)"] & & \Lambda_a(i,j,k) = \mathbb{Z}_D\oplus\mathbb{T}^2 \arrow[d] \\
1 & & 0
\end{tikzcd}$$

where $D = D(i, j, k) = \gcd(p_i, p_j, p_k)$. *Also the restriction map* $\mathrm{Res}_a(i, k) : \mathrm{H}^2(i, k) \mapsto \Lambda_a(i, k)$ *is given by*

$$\begin{tikzcd}
1 \arrow[d] & & 1 \arrow[d] \\
\mathrm{B}^2(i,k) = (2\mathbb{Z})^2 \arrow[d] \arrow[rr, "X_a(i,k)\longrightarrow X_a(i,k)"] & & \mathrm{B}_a(i,k) = (2\mathbb{Z})^2 \arrow[d] \\
\mathrm{Z}^2(i,k) = \mathbb{R}^2 \arrow[d] \arrow[rr, "X_a(i,k)\longrightarrow X_a(i,k)"] & & \mathrm{Z}_a(i,k) = \mathbb{R}^2 \arrow[d] \\
\mathrm{H}^2(i,k) = (\mathbb{R}/2\mathbb{Z})^2 \arrow[d] \arrow[rr, "\mathrm{Res}(i,k)"] & & \Lambda_a(i,k) = (\mathbb{R}/2\mathbb{Z})^2 \arrow[d] \\
1 & & 1
\end{tikzcd}$$

Consequently, we get

$$\Lambda_a(i, j, k)/\mathrm{Res}(i, j, k)\big(\mathrm{H}^2(i, j, k)\big) \cong \mathbb{Z}/(D\mathbb{Z}),$$
$$\Lambda_a(i, k)/\mathrm{Res}(i, k)\big(\mathrm{H}^2(i, k)\big) \cong \{0\}.$$

ii) *The modified* HJR-*map* $\delta : \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) \mapsto \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ *enjoys the following properties*:

a) *The* (i, j, k)*-component and* (i, k)*-component of* $\mathrm{Ker}(\delta)$ *are given by:*

$$\mathrm{Ker}(\delta)_{ijk} = \{0\} \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}),$$
$$\mathrm{Ker}(\delta)_{ik} = (\mathbb{R}/2\mathbb{Z}) \oplus (\mathbb{R}/2\mathbb{Z}) = \Lambda_a(i, k).$$

b) *The image* $\delta([\lambda_a, \mu_a]) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}), a \in \mathrm{Z}_a$, *depends only on the asymmetrization* ASa, *i.e.*,

$$\delta([\lambda_a, \mu_a]) = \delta([\lambda_{\widehat{a}}, 1])$$

where

$$\begin{aligned} \widehat{a}(i, j, k) = (\mathrm{AS}a)(i, j, k) \in \left(\frac{1}{D}\mathbb{Z}\right), \quad i < j < k; \\ \widehat{a}(j, i, k) = \widehat{a}(k, i, j) = \widehat{a}(i, i, k) = \widehat{a}(i, j, j) = \widehat{a}(k, i, k) = 0. \end{aligned} \tag{5.5}$$

c) *Set*

$$\mathrm{Z}_{\widehat{a}} = \{a \in \mathrm{Z}_a : a \text{ satisfies the requirement (5.5)}\}.$$

If $a \in \mathrm{Z}_{\widehat{a}}$, *then the image* $c_a = \delta(\lambda_a, 1) \in \mathrm{Z}^{\mathrm{out}}(G_{\mathrm{m}}, N, \mathbb{T})$ *under the modified* HJR-*map* δ *is in the pull back* $\pi^*_{\mathrm{m}}\big(\mathrm{H}^3(Q, \mathbb{T})\big)$ *and given by*:

$$\begin{aligned} c_a(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) &= c_a(q_1, q_2, q_3) \\ &= \exp\left(2\pi \mathrm{i}\left(\sum_{i<j<k} a(i, j, k)\{e_i(q_1)\}_{p_i}\{e_j(q_2)\}_{p_j}\{e_k(q_3)\}_{p_k}\right)\right) \end{aligned} \tag{5.6}$$

for each $\tilde{q}_1 = (q_1, s_1),\ \tilde{q}_2 = (q_2, s_2),\ \tilde{q}_3 = (q_3, s_3) \in Q_{\mathrm{m}}$.

d) *The modified* HJR-*map* δ_{HJR} *is injective on* Λ_b *and* $\mathrm{Ker}(\delta)$ *is precisely the connected component of* $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$. *If* $b \in \mathrm{Z}_b$, *then*

$$[c_b, \nu_b] = \delta(\lambda_b, 1) \in \mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$$

is given by:

$$c_b(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) = \exp\left(2\pi \mathrm{i}\left(\sum_{i\in\mathbb{N}, j\in\mathbb{N}_0} b(i, j)e_{i,N}(\mathfrak{n}_N(\tilde{q}_2; \tilde{q}_3))\tilde{e}_j(\mathfrak{s}(\tilde{q}_1))\right)\right) \tag{5.7}$$

where

$$\begin{aligned} e_{i,N}(\mathfrak{n}_N(\tilde{q}_2; \tilde{q}_3)) = \frac{\eta_{p_i}\Big(\big[e_i(q_2)\big]_{p_i}; \big[e_i(q_3)\big]_{p_i}\Big)}{p_i}, \\ \tilde{e}_i(\mathfrak{s}(\tilde{q}_1)) = \{e_i(q_1)\}_{p_i} \text{ for } i \geq 1, \quad \tilde{e}_0(\mathfrak{s}(\tilde{q}_1)) = \tilde{e}_0(q_1). \end{aligned} \tag{5.8}$$

The d-part d_{c_b} of c_b is given by ν_b:

$$\left.\begin{aligned} d_{c_b}(q_2;q_3) &= \exp\left(2\pi\mathrm{i}\left(\sum_{j\in\mathbb{N}} b(j,0)\frac{\eta_{p_j}\Big([e_j(q_2)]_{p_j};[e_j(q_3)]_{p_j}\Big)}{p_j}\right)\right)\\ &= \exp\left(2\pi\mathrm{i}\left(\frac{\{\nu_b(\mathfrak{n}_N(q_2;q_3))\}_T}{T}\right)\right),\\ \nu_b(g) &= \pi_T\left(T\sum_{j\in\mathbb{N}} b(j,0)e_{j,N}(g)\right)\in\mathbb{R}/T\mathbb{Z},\quad g\in N, \end{aligned}\right\}\tag{5.9}$$

where $\pi_T : s\in\mathbb{R}\mapsto s_T = s+T\mathbb{Z}\in\mathbb{R}/T\mathbb{Z}$ is the quotient map.

The modular obstruction group $\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})$ looks like the following:

$$\left.\begin{gathered} \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T}) = \mathrm{H}^{\mathrm{out}}_a\oplus\mathrm{H}^{\mathrm{out}}_b,\quad \mathrm{H}^{\mathrm{out}}_b\cong\Lambda_b,\\ \delta([\lambda_a,\mu_a]) = [c_{\mathrm{AS}a}]\in\prod_{i<j<k}\left(\left(\frac{1}{\gcd(p_i,p_j,p_k)}\mathbb{Z}\right)\Big/\mathbb{Z}\right),\ a\in\mathrm{Z}_a,\\ [c_b,\nu_b]=\delta([\lambda_b,1]),\quad \nu_b\in\mathrm{Hom}(N,\mathbb{R}/T\mathbb{Z}),\\ \left[c_b^{i,i}\right] = \Big([p_ib(i,i)-q_ib(i,0)]_{D_i\mathbb{Z}},[-v_ib(i,i)+u_ib(i,0)]_{\mathbb{Z}}\Big)\\ \in\mathbb{Z}\Big/(D_i\mathbb{Z})\oplus\mathbb{R}/\mathbb{Z},\\ \left[c_b^{i,j}\right] = \begin{pmatrix} [m_{i,j}(b(i,j)r_{j,i}+b(j,i)r_{i,j})-n_{i,j}(b(i,0)s_{j,i}+b(j,0)s_{i,j})]_{\mathbb{Z}}\\ [y_{i,j}(b(i,j)r_{j,i}+b(j,i)r_{i,j})+x_{i,j}(b(i,0)s_{j,i}+b(j,0)s_{i,j})]_{\mathbb{Z}}\\ [-b(i,0)w_{i,j}+b(j,0)w_{j,i}]_{\mathbb{Z}} \end{pmatrix}\\ \in\begin{pmatrix}\left(\frac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\end{pmatrix},\quad D(i,j)=\gcd(p_i,p_j,q_i,q_j). \end{gathered}\right\}\tag{5.10}$$

iii) *The map $\partial_{Q_{\mathrm{m}}} : \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})\mapsto\mathrm{H}^3(G,\mathbb{T})$ in the modified* HJR*-exact sequence is given by*:

$$\left.\begin{gathered} \partial_{Q_{\mathrm{m}}}([c_{\widehat{a}}][c_b\nu_b]) = \left[c^G_{\widehat{a}}\right]\in\mathrm{H}^3(G,\mathbb{T}) = X^3(G,\mathbb{T}),\widehat{a}\in\mathrm{Z}_{\widehat{a}}\\ c^G_{\widehat{a}} = \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k}(\mathrm{AS}a)(i,j,k)e_i\otimes e_j\otimes e_k\right)\right),\\ \partial_{Q_{\mathrm{m}}}\big(\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})\big) = \pi_Q^*\big(\mathrm{H}^3(Q,\mathbb{T})\big). \end{gathered}\right\}\tag{5.11}$$

Proof. i) The assertion has been already proven.

ii) For each $i<j<k$, let

$$D(i,j,k) = \gcd(p_i,p_j,p_k)\in\mathbb{Z}.$$

Fix $a \in \mathrm{Z}_a$, i.e., $a \in \mathbb{R}^\Delta$ such that

$$(\mathrm{AS}a)(i,j,k) = a(i,j,k) - a(j,i,k) + a(k,i,j) \in \left(\frac{1}{D(i,j,k)}\mathbb{Z}\right)$$
$$a(i,j,k) = 0 \quad \text{if} \quad j \geq k.$$

Set

$$z_a(i,j,k) = \begin{pmatrix} a(i,j,k) \\ a(j,i,k) \\ a(k,i,j) \end{pmatrix} \in \mathrm{Z}_a = A^{-1}\begin{pmatrix} \left(\frac{1}{D(i,j,k)}\mathbb{Z}\right) \\ \mathbb{R} \\ \mathbb{R} \end{pmatrix}.$$

Then we get

$$Az_a(i,j,k) = \begin{pmatrix} (\mathrm{AS}a)(i,j,k) \\ a(j,i,k) \\ a(k,i,j) \end{pmatrix} \in \begin{pmatrix} \frac{1}{D(i,j,k)}\mathbb{Z} \\ \mathbb{R} \\ \mathbb{R} \end{pmatrix}$$
$$A\mathrm{B}_a(i,j,k) = \mathbb{Z}^3,$$

so that

$$\left[\lambda_a^{i,j,k}, \mu_a^{ijk}\right] \sim \begin{pmatrix} [(\mathrm{AS}a)(i,j,k)]_{\mathbb{Z}} \\ [a(j,i,k)]_{\mathbb{Z}} \\ [a(k,i,j)]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} \left(\frac{1}{D(i,j,k)}\mathbb{Z}\right) \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix}.$$

If $(\mathrm{AS}a)(i,j,k) \in \mathbb{Z}$, the second cocycle μ_a^{ijk} extends to a second cocycle on H which gives $\left(\lambda_a^{i,j,k}, \mu_a^{i,j,k}\right) = \mathrm{Res}\left(\mu_a^{i,j,k}\right)$. Since Range(Res) = Ker(δ), the image $\delta(\lambda_a^{i,j,k}, \mu_a^{i,j,k})$ depends only on the first term $(\mathrm{AS}a)(i,j,k)$ of $Az_a(i,j,k)$. Hence we conclude $\delta([\lambda_a, \mu_a]) = \delta([\lambda_{\widehat{a}}], 1)$. For $\Lambda_a(i,k)$, we have

$$\Lambda_a(i,k) = \mathrm{Res}(i,k)\left(\mathrm{H}^2(i,k)\right),$$

so that the map δ kills the entire $\Lambda_a(i,k)$. This completes the proof of (iia) and (iib).

iic) Set $c_a = \delta(\lambda_a, \mu_a)$ with $a \in \mathrm{Z}_{\widehat{a}}$. We then look at the crossed extension $E_{\lambda_a,\mu_a} \in \mathrm{Xext}(H_\mathrm{m}, L, M, \mathbb{T})$

$$1 \longrightarrow \mathbb{T} \longrightarrow E \underset{\overleftarrow{\ \mathfrak{s}_j\ }}{\xrightarrow{\ j\ }} L \longrightarrow 1.$$

As

$$a(i,j,k) \in \left(\frac{1}{\gcd(p_i,p_j,p_k)}\mathbb{Z}\right) \quad \text{and} \quad e_i(g) \in p_i\mathbb{Z}, \quad g \in L,$$

we have $\mu_a = 1$. Hence observing that $\lambda_a(g;\tilde{h}) = 1$ for every $g \in L \wedge H_\mathrm{m}$ and $\tilde{h} \in H_\mathrm{m}$, we get from (3.15) and (3.16) the following:

$$\begin{aligned} c_a(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) &= \alpha_{\mathfrak{s}(\tilde{q}_1)}(\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3)))\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1;\tilde{q}_2\tilde{q}_3)) \\ &\qquad \times \{\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1;\tilde{q}_2))\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1\tilde{q}_2;\tilde{q}_3))\}^{-1} \\ &= \lambda_a\left(\mathfrak{s}(\tilde{q}_1)\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3)\mathfrak{s}(\tilde{q}_1)^{-1}; \mathfrak{s}(\tilde{q}_1)\right) \\ &= \lambda_a((\mathfrak{s}(\tilde{q}_1) \wedge \mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3))\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3); \mathfrak{s}(\tilde{q}_1)) \end{aligned}$$

$$
\begin{aligned}
&= \lambda_a(\mathfrak{s}(\tilde{q}_1) \wedge \mathfrak{n}_L(\tilde{q}_2; \tilde{q}_3); \mathfrak{s}(\tilde{q}_1))\lambda_a(\mathfrak{n}_L(\tilde{q}_2; \tilde{q}_3); \mathfrak{s}(\tilde{q}_1))\\
&= \lambda_a(\mathfrak{n}_L(\tilde{q}_2; \tilde{q}_3); \mathfrak{s}(\tilde{q}_1))\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k} a(i,j,k)e_{j,k}(\mathfrak{n}_L(\tilde{q}_2; \tilde{q}_3))e_i(\mathfrak{s}(\tilde{q}_1))\right)\right)\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k} a(i,j,k)\{e_i(\tilde{q}_1)\}_{p_i}\{e_j(\tilde{q}_2)\}_{p_j}\{e_k(\tilde{q}_3)\}_{p_k}\right)\right)\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i<j<k} a(i,j,k)\{e_i(q_1)\}_{p_i}\{e_j(q_2)\}_{p_j}\{e_k(q_3)\}_{p_k}\right)\right)\\
&= c_a(q_1; q_2; q_3)
\end{aligned}
$$

for each $\tilde{q}_1 = (q_1, s_1), \tilde{q}_2 = (\tilde{q}_2, s_2), \tilde{q}_3 = (q_3, s_3) \in Q_{\mathrm{m}}$. Thus the assertion (iic) follows.

iid) Since $\mathrm{Res}\big(\mathrm{H}^2(H, \mathbb{T})\big) \cap \Lambda_b = \{0\}$, the modified HJR-map δ is injective on Λ_b. Now fix $b \in \mathrm{Z}_b$. Since $\mu_b = 1$ and $\lambda_b(m; \tilde{h}) = \{1\}$ for every pair $m \in M, \tilde{h} \in H_{\mathrm{m}}$, we have, as in (iic), the following:

$$
\begin{aligned}
c_b(\tilde{q}_1; \tilde{q}_2; \tilde{q}_3) &= \lambda_b(\mathfrak{n}_N(q_2; q_3); \mathfrak{s}(\tilde{q}_1))\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i\in\mathbb{N}, j\in\mathbb{N}_0} b(i,j)e_{i,N}(\mathfrak{n}_N(q_2; q_3))\tilde{e}_j(\mathfrak{s}(\tilde{q}_1))\right)\right)\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i,j\in\mathbb{N}} b(i,j)e_{i,N}(\mathfrak{n}_N(q_2; q_3))e_j(\mathfrak{s}(q_1))\right)\right)\\
&\quad \times \exp\left(2\pi\mathrm{i}\left(\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(\mathfrak{n}_N(q_2; q_3))\tilde{e}_0(\tilde{q}_1)\right)\right)
\end{aligned}
$$

where $e_{i,N}(\mathfrak{n}_N(q_2; q_3))$ is given by (5.8). Also we compute

$$
\begin{aligned}
d_{c_b}(q_2; q_3) = \lambda_b(\mathfrak{n}_N(q_2; q_3); z_0) &= \exp\left(2\pi\mathrm{i}\left(\frac{\nu_b(\mathfrak{n}_N(q_2; q_3))}{T}\right)\right)\\
&= \exp\left(2\pi\mathrm{i}\left(\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(\mathfrak{n}_N(q_2; q_3))\right)\right),\\
\nu_b(g) = \pi_T\left(T\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(g)\right) &\in \mathbb{R}/T\mathbb{Z}, \quad g \in N,
\end{aligned}
$$

with $\pi_T : s \in \mathbb{R} \mapsto s_T = s + T\mathbb{Z} \in \mathbb{R}/T\mathbb{Z}$ the quotient map.

The last assertion, (5.10), on $\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ follows almost automatically from the above computations and Lemma 4.6 in the last section.

iii) We now compute the map

$$\partial_{\pi_{\mathrm{m}}} : \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) \mapsto \mathrm{H}^3(G, \mathbb{T}).$$

We continue to work on the cocycle $(\lambda_{a,b}, 1)$ for $a \in \mathrm{Z}_{\widehat{a}}$ whose restriction to $\{H_{\mathrm{m}}, K\}$ gives rise to the crossed extension $U \in \mathrm{Xext}(H_{\mathrm{m}}, K, \mathbb{T})$:

$$1 \longrightarrow \mathbb{T} \longrightarrow U \underset{\mathfrak{s}_j}{\overset{j}{\rightleftarrows}} K \longrightarrow 1$$

where the group K is given by the following:

$$K = \mathrm{Ker}(\nu_b \circ \pi_G) = \left\{ g \in L : \sum_{i \in \mathbb{N}} b(j, 0) e_{j,N}(g) \in \mathbb{Z} \right\}.$$

Then the following third cocycle $c_G \in \mathrm{Z}^3(G, \mathbb{T})$:

$$\begin{aligned} c_G(g_1; g_2; g_3) &= \alpha_{\mathfrak{s}_H(g_1)}\Big(\mathfrak{s}_j(\mathfrak{n}_M(g_2; g_3))\Big)\mathfrak{s}_j(\mathfrak{n}_M(g_1; g_2g_3)) \\ &\qquad \times \Big(\mathfrak{s}_j(\mathfrak{n}_M(g_1; g_2))\mathfrak{s}_j(\mathfrak{n}_M(g_1g_2; g_3))\Big)^{-1} \\ &= \lambda_{a,b}(\mathfrak{n}_M(g_2; g_3); g_1) = \lambda_a(\mathfrak{n}_M(g_2; g_3); g_1) \\ &= \exp\left(2\pi \mathrm{i} \left(\sum_{i<j<k} a(i, j, k) e_i(g_1) e_j(g_2) e_k(g_3) \right)\right) \\ &= c_a^G(g_1; g_2; g_3), \quad g_1, g_2, g_3 \in G, \end{aligned}$$

is precisely the image $\partial_{\pi_{\mathrm{m}}} \circ \delta(\lambda_{a,b}, 1)$. ♡

§6. Concluding Remark.

The history of cocycle (resp. outer) conjugacy analysis of group actions and group outer actions on an AFD factor goes back to the grand work of Connes, [Cnn3, 4], in the mid 1970's. Since then, the steady progress was accomplished by several hands through the three decades following Connes work, the works of V.F.R. Jone and A. Ocneanu are noteworthy, [Jn, Ocn].

We have now computed the invariants, which determine the outer conjugacy class, of an outer action of a countable discrete abelian group on an AFD factor of type III_λ, $0 < \lambda < 1$. The reduction of outer conjugacy analysis of an outer action of a countable discrete amenable group on an AFD factor of type III_λ down to the associated complete invariants was successfully carried out in our previous work, [KtT1,2,3]. As we have demonstrated in this paper, the computation of invariants is doable as soon as the group in question is specified, except the case of type III_0.

Toward One Parameter Automorphism Group: After the completion of cocycle (resp. outer) conjugacy classification of countable discrete amenable group (resp. outer) actions on an AFD factor, it is only natural to consider the same problem for a continuous group. The first step to this goal is obviously the study of one parameter automorphism group $\{\alpha_t : t \in \mathbb{R}\}$ of an approximately finite dimensional factor $\mathcal{R}_0$ of type II_1. The first steps were already taken by Y. Kawahigashi, [Kw1, 2, 3, 4], who classified, up to cocycle (or stable) conjugacy, the most of one parameter automorphism groups of $\mathcal{R}_0$ constructed from concrete data, which was extended to the case of type III by U-K. Hui, [Hu]. But the general ones with full Connes spectrum are left untouched. One of difficulties is the lack of technique which allows us to create a one cocycle $\{u_s : s \in \mathbb{R}\}$ for a projection $p \in \mathrm{Proj}(\mathcal{R}_0)$ so that the perturbed one parameter automorphism group $\{\mathrm{Ad}(u_t)\circ\alpha_t : t \in \mathbb{R}\}$ leaves the projection p invariant which allows us to localize the analysis of the action. If a projection $p \in \mathrm{Proj}(\mathcal{R}_0)$ is differentiable relative to α, then the derivation δ_α associated with α generates a desired cocycle. But we don't know the answer to the following basic question:

Question: *Does the* C^**-algebra*:

$$A = \left\{x \in \mathcal{R}_0 : \lim_{t\to 0} \|x - \alpha_t(x)\| = 0\right\}$$

contain a non-trivial projection?

If $p \in \mathrm{Proj}(A)$, then for each smooth function $f \in C_c^\infty(\mathbb{R})$ with compact support the element:

$$p(f) = \alpha_f(p) = \int_{\mathbb{R}} f(t)\alpha_t(p)\mathrm{d}t$$

is smooth and one can choose f such a way that $\|p - p(f)\|$ is arbitrarily small so that $\mathrm{Sp}(p(f))$ is concentrated on a neighborhood of the two points $\{0, 1\}$, which allows us to generate a non trivial differentiable projection q near p via contour integral:

$$q = \frac{1}{2\pi\mathrm{i}} \oint_{|z-1|=r} (z - p(f))^{-1}\mathrm{d}z.$$

On the other hand, thanks to the exponential functional calculus, one can generate plenty of differentiable unitaries. For example, if $h \in A_{\mathrm{s.a}}$, then for a real valued smooth function f, we get a differentiable unitary element $\exp(\mathrm{i}f(h))$ of A which can stay near the unitary $\exp(\mathrm{i}h)$ in norm. Hence the group of differentiable unitaries is σ^*-strongly dense in the unitary group $\mathcal{U}(\mathcal{R}_0)$.

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